12.2.65. Prove that the midpoint of the line segment joining \( p(x_1, y_1, z_1) \) & \( q(x_2, y_2, z_3) \) is
\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)
\]

The problem here, as I see it, is that “midpoint” has not been rigorously defined, and even worse the precise definition of a “line segment” has not been developed either. And yet we are asked here to produce a rigorous proof of some result hinging on these notions. This is a bother. The line segment \( \overline{pq} \) in \( \mathbb{R}^3 \) is commonly defined as the following set of points:
\[
\overline{pq} = \{(x_t + t(x_2 - x_1), y_1 + t(y_2 - y_1), z_1 + t(z_2 - z_1)) \mid 0 \leq t \leq 1\}. \tag{1}
\]
There are nicer ways to write (1) using vector notation, but that is best deferred to, say, section 12.5.

The midpoint \( m_{pq} \) of the segment \( \overline{pq} \) is defined to be the point in \( \overline{pq} \) that is “equidistant” from \( p \) and \( q \), meaning the distance between \( m_{pq} \) and \( p \) equals the distance between \( m_{pq} \) and \( q \): \( D(m_{pq}, p) = D(m_{pq}, q) \).

What we have to prove here is \( m = \left( \frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2) \right) \) fits this definition.

That \( m \in \overline{pq} \) is easy to verify: look at the element of \( \overline{pq} \) that we get when we let \( t = 1/2 \):
\[
\left( x_1 + \frac{1}{2}(x_2 - x_1), y_1 + \frac{1}{2}(y_2 - y_1), z_1 + \frac{1}{2}(z_2 - z_1) \right)
\]
This is precisely \( \left( \frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2), \frac{1}{2}(z_1 + z_2) \right) = m \). Done.

Now, to establish equidistance, just take the direct route:
\[
D(m, p) = \sqrt{\left( x_1 - \frac{x_1 + x_2}{2} \right)^2 + \left( y_1 - \frac{y_1 + y_2}{2} \right)^2 + \left( z_1 - \frac{z_1 + z_2}{2} \right)^2}
\]
\[
= \frac{1}{2} \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}
\]
\[
= \frac{1}{2} D(p, q)
\]
\[
D(m, q) = \sqrt{\left( x_2 - \frac{x_1 + x_2}{2} \right)^2 + \left( y_2 - \frac{y_1 + y_2}{2} \right)^2 + \left( z_2 - \frac{z_1 + z_2}{2} \right)^2}
\]
\[
= \frac{1}{2} \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}
\]
\[
= \frac{1}{2} D(p, q)
\]
Therefore \( D(m, p) = D(m, q) \), and so \( m = m_{pq} \). 