1. Let \( f(x) = \sqrt[3]{x} \). The 3rd-order Taylor polynomial centered at 125 for \( f \) is

\[
P_3(x) = f(125) + f'(125)(x - 125) + \frac{f''(125)}{2}(x - 125)^2 + \frac{f'''(125)}{6}(x - 125)^3
\]

\[
= 5 + \frac{1}{3}(125)^{-2/3}(x - 125) - \frac{1}{9}(125)^{-5/3}(x - 125)^2 + \frac{5}{81}(125)^{-8/3}(x - 125)^3
\]

and so

\[
\sqrt[3]{126} = f(126) \approx P_3(126) = 5 + \frac{1}{75} - \frac{1}{228,125} + \frac{1}{6,328,125} \approx 5.0132979358025.
\]

(Note this is very close to the actual value of 5.01329793496458....)

2a. Clearly the series converges when \( x = 0 \). Assuming \( x \neq 0 \), we find that

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^3 x^{4(n+1)}}{(n+1)!} \cdot \frac{n!}{n^3 x^{4n}} \right| = \lim_{n \to \infty} \frac{(n+1)^2 x^4}{n^3} = 0
\]

for all \( x \), and so by the Ratio Test the series converges on \((-\infty, \infty)\). There are no endpoints to consider here.

2b. Since

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^n x^{n+1}}{(n+1)^3} \cdot \frac{n^3}{(-1)^{n-1} x^n} \right| = \lim_{n \to \infty} \frac{n^3 |x|}{(n+1)^3} = |x|
\]

by the Ratio Test the series converges if \( |x| < 1 \), which implies \( x \in (-1, 1) \).

When \( x = 1 \) the series becomes

\[
\sum \frac{(-1)^{n-1}}{n^3},
\]

which converges by the Alternating Series Test. When \( x = -1 \) the series becomes

\[
\sum \frac{(-1)^{2n-1}}{n^3} = - \sum \frac{1}{n^3},
\]

which is a convergent \( p \)-series. The interval of convergence is therefore \([-1, 1]\).

2c. Since

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-2)^n + (x - 1)^n + (x - 1)^n}{\sqrt{n + 1}} \cdot \frac{\sqrt{n}}{(-2)^n(x - 1)^n} \right| = \lim_{n \to \infty} \frac{2|x - 1| \sqrt{n}}{n + 1} = 2|x - 1|,
\]

by the Ratio Test the series converges if \( 2|x - 1| < 1 \), which implies \( x \in \left( \frac{1}{2}, \frac{3}{2} \right) \).

When \( x = 1/2 \) the series becomes

\[
\sum \frac{1}{\sqrt{n}} = \sum \frac{1}{n^{1/4}},
\]
which is a divergent \( p \)-series. When \( x = 3/2 \) the series becomes
\[
\sum (-1)^n \frac{n}{\sqrt{n}},
\]
which converges by the Alternating Series Test. The interval of convergence is therefore \( \left( \frac{1}{2}, \frac{3}{2} \right) \).

3 From the table provided we have
\[
\sum_{n=0}^{\infty} x^n = \frac{1}{1 - x}
\]
for \( |x| < 1 \). Hence
\[
\sum_{n=0}^{\infty} \left( \frac{3}{2x^2 + 1} \right)^n = \frac{1}{1 - \frac{3}{2x^2 + 1}} = \frac{2x^2 + 1}{2x^2 - 2}
\]
for
\[
\left| \frac{3}{2x^2 + 1} \right| < 1 \quad \Rightarrow \quad 2x^2 + 1 > 3 \quad \Rightarrow \quad x^2 > 1 \quad \Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty).
\]
Thus there are two intervals of convergence: \((-\infty, -1)\) and \((1, \infty)\). (Note: the series is not a power series.)

4a The first four terms of the Taylor series for \( f \) centered at 2 are
\[
f(2) + f'(2)(x - 2) + \frac{f''(2)}{2} (x - 2)^2 + \frac{f'''(2)}{6} (x - 2)^3.
\]
Now,
\[
f(x) = \frac{1}{x}, \quad f'(x) = -\frac{1}{x^2}, \quad f''(x) = \frac{2}{x^3}, \quad f'''(x) = -\frac{6}{x^4},
\]
and so the first four terms are
\[
\frac{1}{2} - \frac{1}{4} (x - 2) + \frac{1}{8} (x - 2)^2 - \frac{1}{16} (x - 2)^3.
\]

4b Based on the pattern exhibited by the first five terms, we have
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} (x - 2)^n
\]

5 From the table provided we have \( e^x = \sum_{n=0}^{\infty} x^n / n! \) for all \( x \in (-\infty, \infty) \), and so
\[
e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n}
\]
for all \( x \). Now,
\[
\int \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right) \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} + c
\]
for all \( x \) and arbitrary constant \( c \). Thus, by the Fundamental Theorem of Calculus,
\[
\int_0^{1/2} e^{-x^2} \, dx = \int_0^{1/2} \left( \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n} \right) \, dx = \left[ \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} x^{2n+1} \right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \left( \frac{1}{2} \right)^{2n+1}
\]
We have arrived at an alternating series \( \sum (-1)^n b_n \) with \( b_n = \frac{1}{n!(2n+1)} \left( \frac{1}{2} \right)^{2n+1} \) for \( n \geq 0 \). Evaluating the first few \( b_n \) values,
\[
b_0 = \frac{1}{2}, \quad b_1 = \frac{1}{24}, \quad b_2 = \frac{1}{320}, \quad b_3 = \frac{1}{5376}, \quad b_4 = \frac{1}{110,592}, \quad b_5 = \frac{1}{2703,360}, \quad b_6 = \frac{1}{76,677,120},
\]
we have \( b_6 \approx 1.30 \times 10^{-8} > 10^{-8} \), and \( b_7 \approx 4.05 \times 10^{-10} < 10^{-8} \). By the Alternating Series Estimation Theorem the approximation
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+1)} \left( \frac{1}{3} \right)^{2n+1} \approx b_0 - b_1 + b_2 - b_3 + b_4 - b_5 + b_6 \approx 0.4612810068
\]
will have an absolute error that is less than \( b_7 < 10^{-8} \). Hence the approximation
\[
\int_0^{1/3} e^{-x^2} \, dx \approx \frac{1}{2} - \frac{1}{24} + \frac{1}{320} - \frac{1}{5376} + \frac{1}{110,592} - \frac{1}{2703,360} + \frac{1}{76,677,120} \approx 0.4612810068
\]
has an absolute error less than \( 10^{-8} \).

6 Here \( x = \sqrt{t} - 2 \) implies \( t = (x + 2)^5 \), and so \( y = t + 1 \) gives \( y = (x + 2)^5 + 1 \). Thus we see that
\[
f(x) = (x + 2)^5 + 1,
\]
and from \( t \in [0, 32] \) we see that \( \text{Dom}(f) = [-2, 0] \).

7 There are many possible parametrizations, but one good one is
\[
(x(t), y(t)) = (8, 2)(1 - t) + (-2, -3)t = (8 - 10t, 2 - 5t)
\]
for \( t \in [0, 1] \).

8 It helps to multiply by \( r \) to get
\[
r^2 = 2r \sin \theta + 2r \cos \theta.
\]
Then, since \( x = r \cos \theta, y = r \sin \theta, \) and \( r^2 = x^2 + y^2 \), we obtain
\[
x^2 + y^2 = 2y + 2x.
\]
We can improve on this: from \((x^2 - 2x) + (y^2 - 2y) = 0\) we obtain
\[(x - 1)^2 + (y - 1)^2 = 2,
which is seen to be the equation of a circle centered at \((1, 1)\) with radius \(\sqrt{2}\).

9 Using the identity \(\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)\), area is
\[
\mathcal{A} = \frac{1}{2} \int_0^{2\pi} (2 + \cos \theta)^2 \, d\theta = \frac{1}{2} \int_0^{2\pi} (4 + 4\cos \theta + \cos^2 \theta) \, d\theta
= 2 \int_0^{2\pi} d\theta + 2 \int_0^{2\pi} \cos \theta \, d\theta + \frac{1}{4} \int_0^{2\pi} (1 + \cos 2\theta) \, d\theta
= 4\pi + 2\left[ \sin \theta \right]_0^{2\pi} + \frac{1}{4} \left[ \theta + \frac{1}{2} \sin 2\theta \right]_0^{2\pi}
= 4\pi + 0 + \frac{\pi}{2} = \frac{9\pi}{2}.
\]

Thus concludes the exam.

Hurray.