1a We have
\[
\lim_{n \to \infty} \frac{12n^5 - 4n^2}{3 - 5n - 9n^5} = \lim_{n \to \infty} \frac{12 - 4n^{-3}}{3n^{-5} - 5n^{-4} - 9} = \frac{12 - 4(0)}{3(0) - 5(0) - 9} = -\frac{12}{9} = -\frac{4}{3}.
\]

1b First we evaluate
\[
\lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{1/n} = \lim_{n \to \infty} \exp(\ln n^{1/n}) = \exp(\lim_{n \to \infty} \ln n^{1/n})
\]
\[
= \exp(\lim_{n \to \infty} \frac{\ln n}{n}) = \exp(0) = 1,
\]
where “LR” indicates an application of L’Hôpital’s Rule.

Now, consider the subsequence of \(\{a_n\}_{n=1}^{\infty}\) that consists of the even-indexed terms, which can be denoted by \(\{a_{n_k}\}_{k=1}^{\infty}\) with \(n_k = 2k\) for \(k \geq 1\). Then, using the fact that \(\lim_{n \to \infty} n^{1/n} = 1\), we have
\[
\lim_{k \to \infty} a_{n_k} = \lim_{k \to \infty} (-1)^{n_k} n_k^{1/n_k} = \lim_{k \to \infty} (-1)^{2k} (2k)^{1/(2k)} = \lim_{k \to \infty} (2k)^{1/(2k)} = 1.
\]

Next, consider the subsequence consisting of the odd-indexed terms, which can be denoted by \(\{a_{n_k}\}_{k=1}^{\infty}\) with \(n_k = 2k - 1\) for \(k \geq 1\). Then we have
\[
\lim_{k \to \infty} a_{n_k} = \lim_{k \to \infty} (-1)^{2k-1} (2k - 1)^{1/(2k-1)} = \lim_{k \to \infty} \left[-(2k - 1)^{1/(2k-1)}\right] = -1.
\]

Since \(\{a_n\}\) has two subsequences with different limits, the sequence \(\{a_n\}\) itself cannot converge. That is, \(\{a_n\}\) diverges.

2 Starting by reindexing, we have
\[
\sum_{k=1}^{\infty} 2^{-3k} = \sum_{k=0}^{\infty} 2^{-3(k+1)} = \sum_{k=0}^{\infty} 2^{-3} 2^{-3k} = \sum_{k=0}^{\infty} \frac{1}{8} \left(\frac{1}{8}\right)^k = \frac{1/8}{1 - 1/8} = \frac{1}{7}.
\]

3 For each \(n \geq 1\) we have
\[
s_n = \sum_{k=1}^{n} \left(\frac{1}{k+5} - \frac{1}{k+6}\right)
\]
\[
= \left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{8}\right) + \left(\frac{1}{8} - \frac{1}{9}\right) + \cdots + \left(\frac{1}{n+4} - \frac{1}{n+5}\right) + \left(\frac{1}{n+5} - \frac{1}{n+6}\right)
\]
\[
= \frac{1}{6} - \frac{1}{n+6}.
\]

so
\[
\sum_{k=1}^{\infty} \left(\frac{1}{k+5} - \frac{1}{k+6}\right) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left(\frac{1}{6} - \frac{1}{n+6}\right) = \frac{1}{6}.
\]
4a Since
\[ \lim_{k \to \infty} \frac{k}{\sqrt{k^2 + 25}} = 1 \neq 0, \]
the series diverges by the Divergence Test.

4b Letting \( u = -2x^2 \), we have
\[
\int_1^\infty xe^{-2x^2} \, dx = \lim_{b \to \infty} \int_1^b xe^{-2x^2} \, dx = \lim_{b \to \infty} \int_{-2}^{b} -\frac{1}{4} e^u \, du = \lim_{b \to \infty} -\frac{1}{4} [e^u]_{-2}^{b} = -\frac{1}{4} (0 - e^{-2}) = \frac{e^{-2}}{4},
\]
so the integral
\[
\int_1^\infty xe^{-2x^2} \, dx
\]
converges, and therefore the series converges by the Integral Test.

4c Since
\[
\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = \lim_{k \to \infty} \left| \frac{2^{k+1}}{(k + 1)^{99}} \cdot \frac{k^{99}}{2^k} \right| = \lim_{k \to \infty} 2 \left( \frac{k}{k + 1} \right)^{99} = 2 \left( \lim_{k \to \infty} \frac{k}{k + 1} \right)^{99} = 2(1)^{99} = 2 > 1,
\]
the series diverges by the Ratio Test.

4d Since
\[
\lim_{k \to \infty} |a_k|^{1/k} = \lim_{k \to \infty} \left[ \left( \frac{k}{k + 1} \right)^{2k^2} \right]^{1/k} = \lim_{k \to \infty} \left( \frac{k}{k + 1} \right)^{2k} = e^{-2} \approx 0.1353 < 1,
\]
the series converges by the Root Test.

4e For each \( k \geq 1 \) we have
\[
0 \leq \frac{\sin^2 k}{k\sqrt{k}} \leq \frac{1}{k\sqrt{k}} = \frac{1}{k^{3/2}},
\]
and since \( \sum_{k=1}^{\infty} k^{-3/2} \) is a convergent \( p \)-series, it follows that
\[
\sum_{k=1}^{\infty} \frac{\sin^2 k}{k\sqrt{k}}
\]
converges by the Direct Comparison Test.

4f For each \( k \geq 1 \) we have
\[
0 \leq \frac{k^7}{k^9 + 3} \leq \frac{k^7}{k^9} = \frac{1}{k^2},
\]
and since $\sum_{k=1}^{\infty} k^{-2}$ is a convergent $p$-series, it follows that
\[
\sum_{k=1}^{\infty} \frac{k^7}{k^9 + 3}
\]
converges by the Direct Comparison Test.

5a Since $\ln k$ and $k$ are monotone increasing functions for $k \geq 2$, it follows that
\[
\frac{1}{k \ln^2 k}
\]
is monotone decreasing (i.e. nonincreasing) for $k \geq 2$. Also
\[
\lim_{k \to \infty} \frac{1}{k \ln^2 k} = 0,
\]
and so by the Alternating Series Test the series converges.

5b Since
\[
\lim_{k \to \infty} \left| (-1)^k \left( 1 - \frac{2}{k} \right) \right| = \lim_{k \to \infty} \left( 1 - \frac{2}{k} \right) = 1 \neq 0,
\]
the series diverges by the Divergence Test.