The Definite Integral
The definite integral is a limit of Riemann sums.

Example:
Approximate the definite integral \( \int_{0}^{1} \cos x \, dx \) by a limit of Riemann sums.
I will use a partition where the interval $[0, 1]$ is divided up into 10 equals parts, and I will use the right endpoints.

\[
\int_0^1 \cos x \, dx = \text{area under curve} \approx \\
\frac{1}{10} \cos \left( \frac{1}{10} \right) + \frac{1}{10} \cos \left( \frac{2}{10} \right) + \cdots + \frac{1}{10} \cos \left( \frac{10}{10} \right) = \\
\frac{1}{10} \sum_{i=1}^{10} \cos \left( \frac{i}{10} \right) = 0.817785
\]
We could just as well add up 1 million rectangles.

\[
\frac{1}{10} \sum_{i=1}^{10} \cos\left(\frac{i}{10}\right) = 0.817785
\]

\[
\frac{1}{100} \sum_{i=1}^{100} \cos\left(\frac{i}{100}\right) = 0.839165
\]

\[
\frac{1}{100000} \sum_{i=1}^{100000} \cos\left(\frac{i}{100000}\right) = 0.841469
\]

\[
\int_{0}^{1} \cos x \, dx = \sin x \bigg|_{0}^{1} = \sin 1 - \sin 0 = \sin 1 = 0.841471
\]
If we approximate an integral, and we don’t use many subintervals, then use the midpoints of the subintervals.

Example:

Get an approximation of $\int_{0}^{10} \frac{2}{\sqrt{5 + x^2}} \, dx$ using a Riemann sum with 5 subintervals of width 2.

If we use 5 subintervals of width 2, the partition will be $[0, 2], [2, 4], [4, 6], [6, 8], \text{ and } [8, 10]$.

For best results, select the midpoints which are 1, 3, 5, 7, and 9.
The Riemann Sum Approximating the Integral

\[
\int_{0}^{10} \sqrt[3]{\frac{2}{5 + x^2}} \, dx \approx 2 \times \sqrt[3]{\frac{2}{5 + 1^2}} + 2 \times \sqrt[3]{\frac{2}{5 + 3^2}} + \\
2 \times \sqrt[3]{\frac{2}{5 + 5^2}} + 2 \times \sqrt[3]{\frac{2}{5 + 7^2}} + 2 \times \sqrt[3]{\frac{2}{5 + 9^2}}
\]

\[= 4.48074.\]

In fact \(\int_{0}^{10} \sqrt[3]{\frac{2}{5 + x^2}} \, dx = 4.47866.\)
The Fundamental Theorem of Integral Calculus

Let $F(x)$ be any antiderivative of $f(x)$. Then

$$\int_{a}^{b} f(x) \, dx = F(b) - F(a).$$

Examples:

$$\int_{2}^{6} x^3 \, dx = \left[ \frac{x^4}{4} \right]_{2}^{6} = \frac{6^4}{4} - \frac{2^4}{4} = 320.$$ 

$$\int_{0}^{\pi/3} \sec^2 x \, dx = \tan x \bigg|_{0}^{\pi/3} = \tan \frac{\pi}{3} - \tan 0 = \sqrt{3}.$$
Some Properties of the Definite Integral

\[ \int_{a}^{b} (f(x) + g(x)) \, dx = \int_{a}^{b} f(x) \, dx + \int_{a}^{b} g(x) \, dx \]

\[ \int_{a}^{b} k f(x) \, dx = k \int_{a}^{b} f(x) \, dx \]

\[ \int_{a}^{b} f(x) \, dx = - \int_{b}^{a} f(x) \, dx \]

\[ \int_{a}^{a} f(x) \, dx = 0 \]

\[ \int_{a}^{b} f(x) \, dx = \int_{a}^{c} f(x) \, dx + \int_{c}^{b} f(x) \, dx \]
Examples

\[
\int_{-2}^{5} \left( x^2 + x + 1 \right) \, dx = \frac{x^3}{3} + \frac{x^2}{2} + x \bigg|_{-2}^{5} = \\
\frac{5^3}{3} + \frac{5^2}{2} + 5 - \left( \frac{(-2)^3}{3} + \frac{(-2)^2}{2} + (-2) \right) = 61.833.
\]

\[
\int_{-2}^{5} 10 \left( x^2 + x + 1 \right) \, dx = 10 \int_{-2}^{5} \left( x^2 + x + 1 \right) \, dx = \\
10(61.833) = 618.33.
\]

\[
\int_{5}^{-2} \left( x^2 + x + 1 \right) \, dx = -61.833.
\]
Another Example

Find \( \int_4^{10} \sqrt[3]{x} \, dx \).

\[
\int_4^{10} x^{1/3} \, dx = \frac{x^{4/3}}{4/3} = \frac{3}{4} x^{4/3} \bigg|_4^{10} =
\]

\[
\frac{3}{4} \left( 10^{4/3} - 4^{4/3} \right) = 11.396.
\]
Express as One Integral

- \( \int_0^{10} f(x)\,dx + \int_{10}^{7} f(x)\,dx \)

- \( \int_0^{5} f(x)\,dx + \int_{5}^{15} f(x)\,dx - \int_{20}^{15} f(x)\,dx \)
The Answers

\[ \int_0^{10} f(x) \, dx + \int_{10}^7 f(x) \, dx = \int_0^{10} f(x) \, dx - \int_7^{10} f(x) \, dx = \int_0^7 f(x) \, dx. \]

\[ \int_0^5 f(x) \, dx + \int_5^{15} f(x) \, dx = \int_0^{15} f(x) \, dx. \]

\[ - \int_0^{15} f(x) \, dx = \int_0^{20} f(x) \, dx. \]

Therefore \[ \int_0^5 f(x) \, dx + \int_5^{15} f(x) \, dx - \int_0^{15} f(x) \, dx = \int_0^{15} f(x) \, dx + \int_{15}^{20} f(x) \, dx = \int_0^{20} f(x) \, dx. \]
Finding an Integral without Getting the Antiderivative

Find \( \int_{2}^{10} |x - 4| \, dx \).

The definite integral is equal to the area under the graph of \( y = |x - 4| \) from \( x = 2 \) to \( x = 10 \).
The area is the sum of the areas of the 2 triangles which is $\frac{1}{2}(2)(2) + \frac{1}{2}(6)(6) = 20$.

Therefore $\int_{2}^{10} |x - 4| \, dx = 20$. 
Example- getting the definite integral from the graph

\[ \int_{0}^{6} f(x) \, dx \] where

\[ f(x) = \begin{cases} 
  x - 2 & \text{if } 0 \leq x \leq 3 \\
  2x - 5 & \text{if } 3 < x \leq 6 
\end{cases} \]

Do not get any antiderivatives. Get the answer from the graph.

The definite integral is equal to the area above the \( x \)-axis MINUS the area below the \( x \)-axis.
The definite integral is equal to the area above the x-axis MINUS the area below the x-axis.

Area is equal to \(- \frac{1}{2}(2)(2) + \frac{1}{2}(1)(1) + (3)(1) + \frac{1}{2}(3)(6) = -2 + \frac{1}{2} + 3 + 9 = 10\frac{1}{2}\).
Summations and Definite Integrals

Work is equal to force $\times$ distance.

What happens if force is a variable? Say it's a function of position $x$. For example, say we stretch a spring.

The force exerted by the spring is $F = kx$ where $x$ is the distance the string is stretched from its equilibrium position.
Finding the Work Done in Stretching a Spring

**Ex:** What is the work (energy) done in stretching a spring 5 ft?

We can approximate the work done by breaking up the interval \([0, 5]\) into 5 equal subintervals of length 1 ft. Then we can select an \(x\) (call it \(x_i\)) in each subinterval.
Conclusion of the Spring Example

The work done in stretching the spring the 1st foot is approximately $kx_1 \cdot 1$ where $x_1$ is any number in $[0, 1]$, and the best selection is $\frac{1}{2}$.

An approximation of the total work is

$$W \approx \sum_{i=1}^{5} kx_i \Delta x \text{ where } \Delta x = 1.$$ 

The exact work done is $\int_{0}^{5} kx \, dx$. 
We can use an integral to find the length of a curve.

How can we get the length of a curve given by $y = f(x)$ from $x = a$ to $x = b$?

We can approximate the length by selecting 2 pts and adding up the lengths of the 3 line segments connecting the points.
Finding the length of a Curve

\[ L \approx \sum_{i=1}^{3} \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sum_{i=1}^{3} \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2} \Delta x \]

The exact length is

\[ \int_{a}^{b} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx. \]
Rules for Integration

\[ \int u^n \, du = \frac{u^{n+1}}{n+1} + C ; \quad (n \neq -1) \]

\[ \int \cos u \, du = \sin u + C \quad \int \csc^2 u \, du = -\cot u + C \]

\[ \int \sin u \, du = -\cos u + C \quad \int \sec u \tan u \, du = \sec u + C \]

\[ \int \sec^2 u \, du = \tan u + C \quad \int \csc u \cot u \, du = -\csc u + C \]
Reversing the Chain Rule

Note that
\[ \frac{d}{dx} \left( x^2 + 5 \right)^5 = 5 \left( x^2 + 5 \right)^4 (2x) = 10x \left( x^2 + 5 \right)^4. \]

Now how do we reverse the procedure? That is, how do we determine that
\[ \int 10x \left( x^2 + 5 \right)^4 \, dx = \left( x^2 + 5 \right)^5 + C. \]
Using Substitution

To find \( \int 10x(x^2 + 5)^{4} \, dx \),

let \( u = x^2 + 5 \), \( du = 2x \, dx \), \( dx = \frac{du}{2x} \)

\[
\int 10x(x^2 + 5)^{4} \, dx = 10 \int xu^4 \frac{du}{2x} = \\
10 \int \frac{u^4}{2} \, du = 5 \int u^4 \, du = 5 \frac{u^5}{5} = u^5 \\
= (x^2 + 5)^{5} + C.
\]
Example - Using Substitution

Find \( \int x^3 (x^4 + 6)^{10} \, dx \).

Let \( u = x^4 + 6 \) \( \Rightarrow \) \( du = 4x^3 \, dx \).

\[
\int x^3 (x^4 + 6)^{10} \, dx = \frac{1}{4} \int u^{10} 4x^3 \, dx = \\
\frac{1}{4} \int u^{10} \, du = \frac{1}{4} \frac{u^{11}}{11} = \frac{u^{11}}{44} = \\
\frac{(x^4 + 6)^{11}}{44} + C.
\]
Another Example Using Substitution

Find \( \int \sin^5 6x \cos 6x \, dx \).

Let \( u = \sin 6x \) \( \Rightarrow \) \( du = 6 \cos 6x \, dx \).

\[
\int \sin^5 6x \cos 6x \, dx = \frac{1}{6} \int u^5 \cos 6x \, dx =
\]

\[
\frac{1}{6} \int u^5 \, du = \frac{1}{6} \frac{u^6}{6} = \frac{\sin^6 6x}{36} + C.
\]
Another Example of Substitution

Find $\int t^2 \sin(6t^3) \, dt$.

Let $u = 6t^3 \Rightarrow du = 18t^2 \, dt$.

$$\int t^2 \sin(6t^3) \, dt = \frac{1}{18} \int \sin u \cdot 18 t^2 \, dt$$

$$= \frac{1}{18} \int \sin u \, du = \frac{1}{18} \cos u = -\frac{1}{18} \cos(6t^3) + C.$$
Be Careful

\[ \int \left( x^4 + 1 \right)^3 \, dx \text{ can't be solved using } \]

the substitution \( u = x^4 + 1 \).

If we let \( u = x^4 + 1 \) then \( du = 4x^3 \, dx \). We can not do the following:

\[ \int \left( x^4 + 1 \right)^3 \, dx = \frac{1}{4x^3} \int u^3 4x^3 \, dx = \frac{1}{4x^3} \int u^3 \, du \]

This is incorrect
We have to expand.

Note that \( \int \left( x^4 + 1 \right)^3 \, dx \neq \frac{(x^4 + 1)^4}{4} + C \).

To find the integral, expand.

\[
\left( x^4 + 1 \right)^3 = (x^4 + 1)(x^8 + 2x^4 + 1) = x^{12} + 3x^8 + 3x^4 + 1.
\]

\[
\int \left( x^4 + 1 \right)^3 \, dx = \int \left( x^{12} + 3x^8 + 3x^4 + 1 \right) \, dx = \frac{x^{13}}{13} + \frac{x^9}{3} + \frac{3x^5}{5} + x + C.
\]
More Examples

Find \[ \int_0^4 \sqrt{3x + 9} \, dx. \]

Let \( u = 3x + 9 \Rightarrow du = 3 \, dx. \)

\[
\int_0^4 \sqrt{3x + 9} \, dx = \frac{1}{3} \int_{x=0}^{x=4} u^{1/2} \, 3 \, dx =
\frac{1}{3} \int_{x=0}^{x=4} u^{1/2} \, du =
\frac{1}{3} \left[ \frac{u^{3/2}}{3/2} \right]_0^4 =
\frac{2}{9} \left( (3x + 9)^{3/2} \right)_0^4 =
\frac{2}{9} \left( 21^{3/2} - 9^{3/2} \right) = 15.3854.
\]
A Harder Problem

Find \[ \int \frac{30x^2 + 200x}{\sqrt[3]{x^3 + 10x^2 + 2}} \, dx \, . \]

If we let \( u = x^3 + 10x^2 + 2 \), will this work out?
The substitution will work out.

Find \[
\int \frac{30x^2 + 200x}{\sqrt[3]{x^3 + 10x^2 + 2}} \, dx.
\]

\[u = x^3 + 10x^2 + 2 \implies du = \left(3x^2 + 20x\right)dx.\]

\[
\int \frac{30x^2 + 200x}{\sqrt[3]{x^3 + 10x^2 + 2}} \, dx = 10 \int \frac{3x^2 + 20x}{u^{1/3}} \, dx =
\]

\[10 \int u^{-1/3} \, du = 10 \frac{u^{2/3}}{2/3} = 10 \cdot \frac{3}{2} u^{2/3} =
\]

\[15 \left(x^3 + 10x^2 + 2\right)^{2/3} + C.\]