Section 3.5- Modeling and Optimization

13. Two sides of a triangle have lengths a and b, and the angle between them is $\theta$. What value of $\theta$ will maximize the triangle's area? (Hint: $A = \frac{1}{2}ab \sin \theta$)

Note that a and b are constants.
$A$ will be a maximum when $A' = 0$.
$A' = \frac{1}{2}ab \cos \theta = 0$ when $\cos \theta = 0$.
The only feasible value of $\theta$ where this occurs is $\theta = 90^\circ$.
Therefore the maximum area will occur when the triangle is a right triangle.

19. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

Let $x$ and $y$ be as shown in the diagram below:

The radius of the cylinder is $x$ and its height is $2y$.
The volume of a cylinder is
$V = \pi r^2 h = \pi x^2 (2y)$
Get $V$ as function of just one variable.
$x^2 + y^2 = 100 \Rightarrow x^2 = 100 - y^2$
Therefore
\[ V = \pi(100 - y^2)(2y) = 200\pi y - 2\pi y^3 \]

V will be a maximum when \( V' = 0 \).
\[ V' = 200\pi - 6\pi y^2 = 0 \Rightarrow 200\pi = 6\pi y^2 \Rightarrow y^2 = \frac{200\pi}{6\pi} = \frac{100}{3} \Rightarrow y = \sqrt{\frac{100}{3}} \approx 5.7735. \]
\[ x^2 = 100 - \frac{100}{3} = \frac{200}{3} \Rightarrow x = \sqrt{\frac{200}{3}} \approx 8.1650. \]
The maximum volume will occur when the radius is \( 8.165 \text{ cm} \) and the height is \( 2(5.7735) = 11.547 \text{ cm} \).
The maximum volume is \( \pi r^2 h = \pi (8.165)^2 (11.547) = 2418 \text{ cm}^3 \).

39. At noon, ship A was 12 nautical miles due north of ship B. Ship A was sailing south at 12 knots (nautical miles per hour) and continued to do so all day. Ship B was sailing east at 8 knots and continued to do so all day.
a) Start counting time with \( t = 0 \) at noon and express the distance \( s \) between the ships as a function of time.

Use \( D = RT \).

In \( t \) hours, ship A will go \( 12t \) miles south and ship B will go \( 8t \) miles east. So after \( t \) hours we have the following:

The distance \( s \) between the 2 ships is \( \sqrt{(8t)^2 + (12 - 12t)^2} \).
\[ s' = \frac{1}{2} \left(208 t^2 - 288 t + 144\right)^{-1/2} (416 t - 288) = \frac{416 t - 288}{2 \sqrt{208 t^2 - 288 t + 144}} \]

At noon, \( t = 0 \) and
\[ s' = -\frac{288}{2 \sqrt{144}} = -\frac{288}{24} = -12 \text{ knots} . \]

At 1:00, \( t = 1 \) and
\[ s' = \frac{416 - 288}{2 \sqrt{208 - 288 + 144}} = \frac{128}{16} = 8 \text{ knots} . \]

c) The visibility that day was 5 nautical miles. Did the ships ever sight each other?

To answer this question we need to find the minimum value of \( s \). This will occur when \( s' = 0 \).

\[ s' = 0 \text{ when the numerator } 416 t - 288 = 0 \Rightarrow t = 288/416 \approx 0.6923 \text{ hours} . \]

At this time
\[ s = \sqrt{208 \times (0.6923)^2 - 288 \times (0.6923) + 144} = 6.656 \text{ nautical miles.} \]

This is the closest distance between the 2 ships, and consequently the 2 ships did not sight each other.

d) Graph \( s \) and \( s' \) together as functions of \( t \) for \(-1 \leq t \leq 3\).

Compare the graphs and reconcile what you see with your answers in parts b and c.
When $s' = 0$, $s$ is a minimum.

The main thing to observe is that the minimum $s$ will occur when $s' = 0$.

e) The graph of $s'$ looks as if it has a horizontal asymptote in the 1st quadrant. This suggests that $s'$ has a limiting value as $t \to \infty$. What is this limiting value? What is it's relation to the ships' individual speeds?

To find $\lim_{t \to \infty} \frac{416t - 288}{2 \sqrt{208} t^2 - 288t + 144}$ ignore all terms on the top and bottom expect for the terms of highest degree.

$$\lim_{t \to \infty} \frac{416t}{2 \sqrt{208} t^2} = \lim_{t \to \infty} \frac{416t}{2 \left( \sqrt{208} \right)} = \frac{416}{2 \sqrt{208}} \approx 14.42.$$  

This means that the graph of $s'$ has a horizontal asymptote $y = 14.42$

Note that $\sqrt{12^2 + 8^2} = \sqrt{144 + 64} = 14.42$. This is the square root of the sum of the squares of the 2 ship speeds.
43. It costs you $c$ dollars to manufacture and distribute backpacks. If the backpacks sell at $x$ dollars each, the number sold is

$$n = \frac{a}{x - c} + b(100 - x)$$

where $a$ and $b$ are positive constants. What selling price will bring the maximum profit?

For each packpack sold they make a profit of $(x - c)$. The total profit is

$$P(x) = n(x)(x - c) = \frac{a}{x - c}(x - c) + b(100 - x)(x - c)$$

$$= a + b(100 - x)(x - c).$$

$$P'(x) = b(-1)(x - c) + b(100 - x)(1) =$$

$$-bx + bc + 100b - bx - x = -2bx + bc + 100b = 0$$

$$\Rightarrow \quad bc + 100b = 2bx \quad \Rightarrow \quad x = \frac{bc + 100b}{2b} = \frac{c}{2} + 50.$$