Calculus (6th edition) by James Stewart

Section 12.5- Alternating Series

7. Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{3n-1}{2n+1}$ for convergence or divergence.

If $\lim_{n\to\infty} |a_n| \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

$$\lim_{n\to\infty}\left|(-1)^n \frac{3n-1}{2n+1}\right| = \lim_{n\to\infty}\frac{3n-1}{2n+1} = \lim_{n\to\infty}\frac{3n}{2n} = \frac{3}{2} \neq 0.$$

Hence the series diverges.

15. Test the series $\sum_{n=1}^{\infty}\frac{cos\, n\pi}{n^{3/4}}$ for convergence.

The series is an alternating series because
$cos\, 1\pi = -1,\ cos\, 2\pi = 1,\ cos\, 3\pi = -1,\ cos\, 4\pi = 1,$
and so forth.

Therefore we can express the series as $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^{3/4}}$.

The terms $\frac{1}{n^{3/4}}$ are positive, they are decreasing, and
$\lim_{n\to\infty}\frac{1}{n^{3/4}} = 0$. Therefore the series converges.

19. Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^n}{n!}$ for convergence or divergence.

$$\sum_{n=1}^{\infty}(-1)^n\frac{n^n}{n!} = -1 + 2 - 4.5 + 10.6667 - \cdots$$

This series diverges because the terms do not converge to 0.
In fact $|a_n| \to \infty$. Here is proof that $|a_n| \geq 1,$

$$\frac{n^n}{n!} = \frac{n \times n \times \cdots \times n}{n \times (n-1) \times \cdots \times 1} \geq 1$$

Since the terms do not converge to 0, the series diverges.

25. Show that the series $\sum_{n=0}^{\infty}\frac{(-1)^n}{10^n n!}$ is convergent. How many terms of the series do we need to add in order to find the sum accurate to within 0.000005?

The series converges because it is an alternating series, the terms are decreasing, and

$$\lim_{n\to\infty}\frac{1}{10^n n!} = 0.$$

If we sum up $\sum_{n=0}^{k}\frac{(-1)^n}{10^n n!}$, this will be within $\frac{1^{k+1}}{10^{k+1}(k+1)!}$ of the sum of the whole series.

So the question is, when is $\frac{1}{10^{k+1}(k+1)!} < 0.000001$?

If $f(k) = \frac{1}{10^{k+1}(k+1)!}$ then

$f(1) = 5\cdot 10^{-3}$

$f(2) = 1.6666666666\cdot10^{-4}$

$f(3) = 4.1666666666\cdot10^{-6}$

$f(3) < 0.000001$ when k is 3 .
Note that when $k = 3$, there are 4 terms.

Hence, to get this accuracy we can add just the 1st 4 terms.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{10^n n!} \approx \sum_{n=0}^{3} \frac{(-1)^n}{10^n n!} \approx 0.904833$$

29. Approximate the sum $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2}{10^n}$ correct to four decimal places.

$\displaystyle\sum_{n=1}^{k} \frac{(-1)^{n-1} n^2}{10^n}$ is correct to within $\displaystyle\frac{(k+1)^2}{10^{k+1}}$.

If we want accuracy to 4 decimal places our sum must be within 0.00005 of the actual sum.

If $f(k) = \dfrac{(k+1)^2}{10^{k+1}}$ then

$$f(3) = 1.6 \cdot 10^{-3}$$

$$f(4) = 2.5 \cdot 10^{-4}$$

$$f(5) = 3.6 \cdot 10^{-5}$$

$$f(6) = 4.9 \cdot 10^{-6}$$

$f(5) = 0.000036$ is small enough. Therefore we can sum 5 terms and

$$\sum_{n=1}^{5} \frac{(-1)^{n-1} n^2}{10^n} = 0.0676 \blacksquare$$