Calculus (6th edition) by James Stewart

Section 12.3- Integral Test

7. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} ne^{-n}$ is convergent or divergent.

Check $\int_1^{\infty} xe^{-x}dx$.

Let $u = x$ and $dv = e^{-x}$.

Then $du = 1$ and $v = -e^{-x}$.

$$\int xe^{-x} = -xe^{-x} - \int(-e^{-x})dx =$$

$$-xe^{-x} + \frac{e^{-x}}{-1} = -xe^{-x} - e^{-x} =$$

$$-e^{-x}(x+1)\,.$$

$$\int_1^N xe^{-x}dx = -e^{-x}(x+1)\Big]_1^N = e^{-x}(x+1)\Big]_N^1 =$$

$$e^{-1}(2) - e^{-N}(N+1) = \frac{2}{e} - \frac{N+1}{e^N}\,.$$

$$\lim_{N\to\infty}\frac{N+1}{e^N} = \lim_{N\to\infty}\frac{1}{e^N} = 0.$$

Therefore $\int_1^{\infty} xe^{-x}dx = \frac{2}{e}$ and the series $\sum_{n=1}^{\infty} ne^{-n}$ converges.

13. Determine whether the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \cdots$$

is convergent or divergent.

The general term is $\frac{1}{2n-1}$ where $n = 1,\ 2,\ 3,\ 4,\ \ldots$.

Check the integral $\int_1^{\infty} \frac{1}{2x-1} dx = \frac{1}{2} ln(2x-1)\Big]_1^{\infty} =$

$$\frac{1}{2} ln(2 \cdot \infty - 1) - \frac{1}{2} ln 1 = \infty \, .$$

Therefore the series diverges.

17. Use the Integral Test to determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2+4}$ is convergent or divergent.

$$\int_1^{\infty} \frac{dx}{x^2+4} = \frac{1}{2} tan^{-1} \frac{x}{2}\Big]_1^{\infty} = \frac{1}{2}\left(tan^{-1} \infty - tan^{-1} \frac{1}{2} \right) \approx$$

$$\frac{1}{2}\left(\frac{\pi}{2} - 0.46365 \right) = 0.5536 < \infty \, .$$

Note: it is more rigorous actually to look at the limit $\lim_{N \to \infty} tan^{-1} N$ which is $\frac{\pi}{2}$.

Therefore the integral and the series converge.

21. Use the Integral Test to determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \, ln \, n}$ is convergent or divergent.

$\int_2^{\infty} \frac{dx}{x \, ln \, x} = \int_{x=2}^{x=\infty} \frac{du}{u}$ where $u = ln \, x$ and $du = \frac{1}{x} dx.$

$$\int_{x=2}^{x=\infty} \frac{du}{u} = ln \, |u| = ln(ln \, x)\Big]_2^{\infty} = ln(ln \, \infty - ln \, 2) = ln \, \infty = \infty \, .$$

Since the integral is divergent, the series is divergent.

27. Find the values of p for which the series $\sum_{n=2}^{\infty} \frac{1}{n(ln\, n)^p}$ is convergent.

From problem 21, we know that if $p = 1$, the series diverges.

For $p \neq 1$,

$$\int \frac{dx}{x(ln\, x)^p} = \int \frac{du}{u^p} = \int u^{-p} du = \frac{u^{1-p}}{1-p} \text{ where } u = ln\, x.$$

$$\int_2^{\infty} \frac{dx}{x(ln\, x)^p} = \frac{(ln\, x)^{1-p}}{1-p}\Bigg]_2^{\infty} = \frac{(\infty)^{\mathbf{1-p}} - (ln\, 2)^{1-p}}{1-p}.$$

The integral will equal ∞ if ∞ is raised to a positive power.
On the other hand, ∞ raised to a negative power is 0.
Therefore the integral will converge if $1 - p < 0 \quad \Rightarrow$
$1 < p \quad \Rightarrow \quad p > 1.$

The integral will converge if $p > 1$.