Calculus (6th edition) by James Stewart

Section 3.8- Related Rates

9. If \( z^2 = x^2 + y^2 \), \( dx/dt = 2 \) and \( dy/dt = 3 \), find \( dz/dt \) when \( x = 5 \) and \( y = 12 \).

Differentiate both sides with respect to \( t \).

Remember that \( \frac{d}{dt} z^2 = 2z \frac{dz}{dt} = 2zz' \), and similarly

\[
\frac{d}{dt} x^2 = 2xx' \quad \text{and} \quad \frac{d}{dt} y^2 = 2yy'.
\]

So we get \( 2zz' = 2xx' + 2yy' \). Solve for \( z' \).

\[
z' = \frac{2xx' + 2yy'}{2z} = \frac{xx' + yy'}{z}.
\]

The only thing we don't know is \( z \).

But \( z^2 = x^2 + y^2 \) \( \Rightarrow \) \( z^2 = 5^2 + 12^2 = 169 \) \( \Rightarrow \)

\[
z = \pm \sqrt{169} = \pm 13.
\]

Therefore \( z = \frac{xx' + yy'}{z} = \frac{(5)(2) + (12)(3)}{\pm 13} = \pm \frac{46}{13} \).

13. A street light is mounted at the top of a 15 ft tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/sec along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

Look at the diagram which consists of a smaller right triangle inside a bigger right triangle. \( x \) is the distance of the street light from the tip of the shadow and \( y \) is the distance of the man from the street light.
The rate at which the tip of the shadow is moving is \( \frac{dx}{dt} \).

This is the unknown in the problem.

We know that \( \frac{dy}{dt} = 5 \text{ ft/sec} \).

We can get a relationship between \( x \) and \( y \) by using the fact that the 2 right triangles are similar. Note that the base of the smaller triangle is \( x - y \), and the relationship is
\[
\frac{15}{x} = \frac{6}{x - y} \quad \Rightarrow \quad 15x - 15y = 6x \quad \Rightarrow \quad 9x = 15y
\]
\[
\Rightarrow \quad x = \frac{5}{3}y.
\]

Therefore \( \frac{dx}{dt} = \frac{dx}{dy} \cdot \frac{dy}{dt} = \frac{5}{3} \cdot 5 = \frac{25}{3} \text{ ft/sec} \).

15. Two cars start moving from the same point. One travels south at 60 m/hr and the other travels west at 25 m/hr. At what rate is the distance between the cars increasing two hours later?

Look at the situation \( t \) hrs later, where \( t \) is an arbitrary time.

The 1st car has moved south (say a distance \( x \)) and the 2nd car has...
moved west (say a distance \( x \)). We have the triangle below.

\[
x^2 + y^2 = s^2
\]

\[\text{differentiate both sides with respect to } t\]

\[
2xx' + 2yy' = 2ss' \Rightarrow s' = \frac{xx' + yy'}{s}
\]

where \( x' = 25 \) and \( y' = 60 \) and \( s' \) is the unknown in the problem. Note that 2 hours later, \( x = 50 \text{ m}, \ y = 120 \text{ m}, \) and \( s = \sqrt{50^2 + 120^2} = 13 \text{ m}. \) Therefore

\[
s' = \frac{(50)(25) + (120)(60)}{13} = 65 \text{ m/hr.}
\]

17. A man starts walking north at 4 ft/s from a point P. Five minutes later a woman starts walking south at 5 ft/s from a point 500 ft due east of P. At what rate are the people moving apart 15 minutes after the woman starts walking?

5 minutes = 300 seconds, and after 5 minutes the man has traveled \( 4 \times 300 = 1200 \text{ ft.} \) Look at the diagrams below.
This is at the point when the woman starts walking south.

\[ s^2 = y^2 + 500^2 \implies s' = \frac{yy'}{s}. \]

15 minutes is \(15 \times 60 = 900\) s.

\[ y = 1200 + 9t \implies y' = 9 \] and at the instant, \(y = 1200 + 9(900) = 9300\) ft.

Therefore \(s' = \frac{(9300)(9)}{\sqrt{500^2 + 9300^2}} = 8.987\) ft/s.

25. A water trough is 10 m long and a cross-section has the shape of an isosceles trapezoid that is 30 cm wide at the bottom, 80 cm wide at the top. If the trough is being filled with water at a rate of 0.2 m³/min, how fast is the water level rising when the water is 30 cm deep?

If the top is 80 cm wide and the bottom is 30 cm, then \(80 - 30 = 50\) and the top of the 2 triangles shown below must have length 25 cm.
The volume of water consists of 2 parts, one of which is a rectangular solid.

To get the volume of water note that the rectangular solid has volume $30 \times h \times 1000 = 30,000h$. The other 2 parts of the volume each have a volume of $\frac{1}{2}rh \times 1000$.

Therefore $V$ (water) $= 30,000h + 2 \cdot \frac{1}{2}rh \cdot 1000$

$V = 30,000h + 2 \cdot \frac{1}{2}rh \cdot 1000 = 30,000h + \left( \frac{h}{2} \right) h 1000 = 30,000h + 500h^2$

$V' = 30,000h' + 1000hh' = \frac{2 m^3}{min} = \frac{2 \times 10^6 cm^3}{min}$

$h'(30,000 + 1000h) = 20 \times 10^4 \Rightarrow$

$h' = \frac{20 \times 10^4}{30,000 + 1000 \times 30} = \frac{20 \times 10^4}{6 \times 10^4} = \frac{10}{3} cm/m$

29. Two sides of a triangle have lengths 4 m and 5 m. The angle between them is increasing at a rate of 0.06 rad/sec. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$. 
The area of a triangle is $A = \frac{1}{2}bh$ where $b$ is the base and $h$ is the height.

\[ h = \frac{4}{4} = \sin \theta \Rightarrow h = 4 \sin \theta \]

So $A = \frac{1}{2}(5)(4\sin \theta) = 10 \sin \theta$.

\[ \frac{dA}{dt} = \frac{dA}{d\theta} \cdot \frac{d\theta}{dt} = 10 \cos \theta \frac{d\theta}{dt} = 10 \left(\cos \frac{\pi}{3}\right)(0.06) = 10 \left(\frac{1}{2}\right)(0.06) = 0.3 \text{ m}^2/\text{sec} \]

41. A plane flying at constant speed of 300 km/hr passes over a ground radar station an altitude of 1 km and climbs at an angle of 30°. At what rate is the distance from the plane to the radar station increasing a minute later?
The unknown in this problem is \( \frac{ds}{dt} \).

We know the speed of the plane which is \( \frac{dx}{dt} = 300 \) km/hr.

The obtuse angle in the triangle is 120°, and we can use the Law of Cosines.

\[
\begin{align*}
\cos 120^\circ &= \frac{x^2 + 1^2 - s^2}{2(1)(x)} \\
\cos 120^\circ &= \frac{x^2 + 1 - 2x}{-1/2} \\
\cos 120^\circ &= \frac{x^2 + 1 + x}{2}
\end{align*}
\]

Differentiate this equation with respect to \( t \).

\[
2ss' = 2xx' + x'
\]

\[
s' = \frac{2xx' + x'}{2s}
\]

We know \( x' = 300 \) and we can find \( x \) and \( s \) at the instant of time we want which is 1 minute later. Remember that 1 minute is 1/60 hr.

Since \( D = RT \), \( x = (300)(1/60) = 5 \) miles.

\[
s^2 = x^2 + 1 + x \Rightarrow s^2 = (5)^2 + 1 + 5 = 31
\]

\[
\Rightarrow s = \sqrt{31} \text{ miles.}
\]

Therefore

\[
s' = \frac{2xx' + x'}{2s} = \frac{2(5)(300) + 300}{2\sqrt{31}} = \frac{3300}{2\sqrt{31}} = \frac{1650}{\sqrt{31}} \approx 296 \text{ km/hr}
\]
43. A runner sprints around a circular track of radius 100 m at a constant speed of 7 m/sec. The runner's friend is standing at a distance of 200 m from the center of the track. How fast is the distance between them changing when the distance between them is 200 m?

Using the Law of Cosines,
\[ x^2 = r^2 + x^2 - 2(r)(x)\cos \theta \]

The unknown in this problem is \( \frac{dx}{dt} \).

Differentiate with respect to \( t \)
\[
2x \frac{dx}{dt} = -40,000(-\sin \theta) \frac{d\theta}{dt}
\]

\[
\frac{dx}{dt} = \frac{40,000 \sin \theta}{2x} \frac{d\theta}{dt} = \frac{20,000 \sin \theta}{x} \frac{d\theta}{dt}
\]

At the instant we want, \( x = 200 \text{ m} \).

We need \( \frac{d\theta}{dt} \). We must convert 7 m/sec to rad/sec.

We use the fact that 1 complete revolution is \( 2\pi r = 2\pi(100) = 200\pi \text{ m} \) and 1 revolution is \( 2\pi \) radians.

\[
\frac{7 \text{ m}}{\text{sec}} \cdot \frac{1 \text{ revolution}}{200 \pi \text{ m}} \cdot \frac{2\pi \text{ rad}}{1 \text{ revolution}} = \frac{7 \text{ rad}}{100 \text{ sec}}
\]
So at the instant we want, \( \frac{d\theta}{dt} = \frac{7}{100} \text{ rad/sec} \).

We also need \( \sin \theta \). We can get \( \cos \theta \).

\[
x^2 = 100^2 + 200^2 - 40,000 \cos \theta \Rightarrow \\
200^2 = 100^2 + 200^2 - 40,000 \cos \theta \Rightarrow \\
\cos \theta = \frac{10,000}{40,000} = \frac{1}{4}.
\]

To find \( \sin \theta \), consider the right triangle below.

\[
\begin{align*}
&\theta \\
\hline
&4 \\
&.1
\end{align*}
\]

\[
h = \sqrt{16 - 1} = \sqrt{15} \Rightarrow \sin \theta = \frac{\sqrt{15}}{4}.
\]

\[
\frac{dx}{dt} = \frac{20,000 \sin \theta \, d\theta}{x} = \frac{20,000 \cdot \sqrt{15}}{4} \cdot \frac{7}{100} = \frac{7\sqrt{15}}{4} \text{ m/sec}
\]