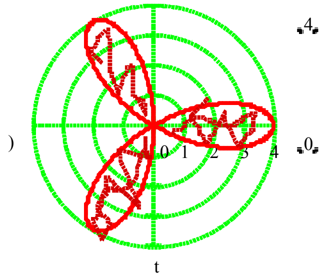
Calculus III (8th edition)
by Larson, Hostetler, and Edwards

Section 14.3- Change of Variables: Polar Coordinates

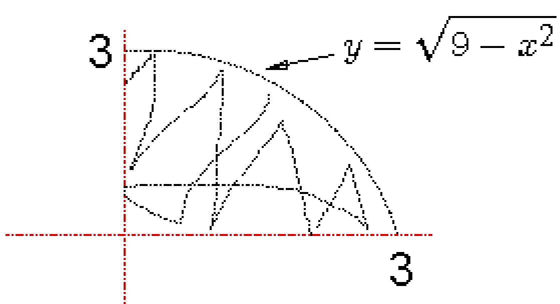
8. Use polar coordinates to describe the region



The region is given by $R = \{(r,\theta) : r = 4cos\,3\theta,\ 0 \le \theta \le 2\pi\}$.

17. Evaluate the iterated integral $\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)^{3/2}\,dy\,dx$ by converting to polar coordinates.

The region R goes from $y=0$ to $y=\sqrt{9-x^2}$ (top half of circle of radius 3) and x goes from 0 to 3. The region R is



$r^2 = x^2 + y^2 \quad \Rightarrow \quad (x^2+y^2)^{3/2} = (r^2)^{3/2} = r^3.$

Note: Don't forget the extra factor of r!

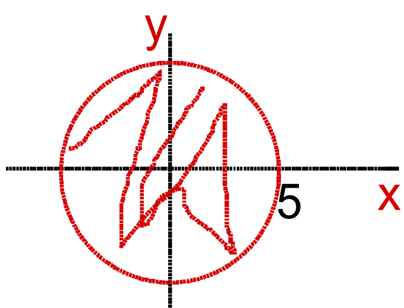
$$\int_0^3 \int_0^{\sqrt{9-x^2}} (x^2+y^2)^{3/2}\,dy\,dx = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=3} r^3 \cdot \boldsymbol{r}\,dr\,d\theta =$$

(this is better) $\displaystyle\int_{r=0}^{r=3}\int_{\theta=0}^{\theta=\pi/2} r^4 d\theta dr = \int_{r=0}^{r=3} r^4\theta\Big]_0^{\pi/2} dr =$

$$\int_0^3 r^4\frac{\pi}{2}dr = \frac{\pi}{2}\cdot\frac{r^5}{5} = \frac{\pi r^5}{10}\Big]_0^3 = \frac{243\pi}{10}\,.$$

29. Use a double integral in polar coordinates to find the volume of the solid bounded by
$z = \sqrt{x^2+y^2}\,,\ z = 0,\ x^2+y^2 = 25.$

The region is given by $x^2+y^2 \le 25\,.$



The volume of the solid is $\displaystyle 4\int_{x=0}^{x=5}\int_{y=0}^{\sqrt{25-x^2}} \sqrt{x^2+y^2}dydx.$
I am taking advantage of symmetry.

Since $z = \sqrt{x^2+y^2} = r$, the volume is also given by

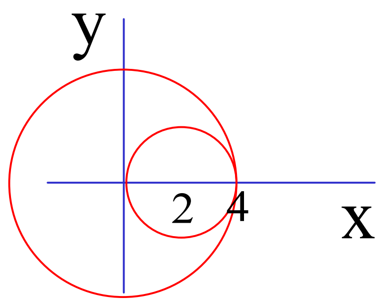
$$4\int_{r=0}^{r=5}\int_{\theta=0}^{\theta=\pi/2} r\cdot \boldsymbol{r} d\theta dr = 4\int_{r=0}^{r=5} r^2\theta\Big]_0^{\pi/2} dr = 4\int_0^5 \frac{r^2\pi}{2}dr =$$

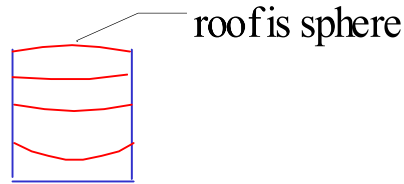
$$\frac{2\pi r^3}{3}\Big]_0^5 = \frac{2\pi(125)}{3} = \frac{250\pi}{3}\,.$$

31. Use a double integral in polar coordinates to find the volume of the solid bounded by the curves:

Inside hemisphere $z = \sqrt{16 - x^2 - y^2}$ and inside the cylinder $x^2 + y^2 - 4x = 0$.

Look at the bottom of the solid which is on the xy-plane. $x^2 + y^2 - 4x = 0 \ \equiv \ (x-2)^2 + y^2 = 4$ which is a circle of radius 2 centered at $(2, 0)$.

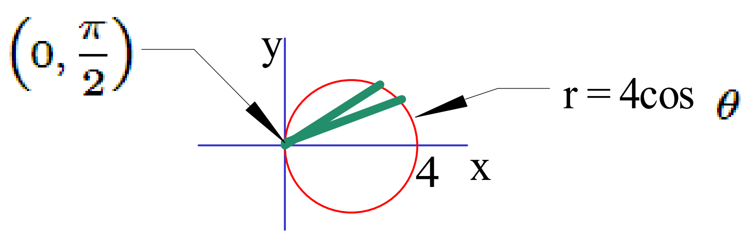




$$x^2 + y^2 - 4x = 0 \equiv r^2 - 4rcos\,\theta = 0 \equiv r(r - rcos\,\theta) = 0$$
$$\equiv r = rcos\,\theta.$$

$z = \sqrt{16 - x^2 - y^2} \equiv z = \sqrt{16 - r^2}$
The region R is $(x-2)^2 + y^2 = 4,$ the circle of radius 2 on the xy-plane. The "roof" has a height of $z = \sqrt{16 - x^2 - y^2}$.

The volume is given by $\displaystyle\iint_R \sqrt{16 - x^2 - y^2}\, dA =$



$$2\int_0^{\pi/2}\int_0^{4cos\,\theta} \sqrt{16 - r^2}\, r dr\, d\theta =$$

$$\frac{2}{-2}\int_0^{\pi/2}\int_0^{4cos\,\theta}\sqrt{16-r^2}\,(-2rdr)d\theta$$

$$(u=16-r^2,\ du=-2rdr)$$

$$=-\int_0^{\pi/2}\int_0^{r=4cos\,\theta}u^{1/2}dud\theta=$$

$$-\int_0^{\pi/2}\int_0^{r=4cos\,\theta}\frac{2}{3}u^{3/2}\Big]_{r=0}^{r=4cos\,\theta}d\theta=$$

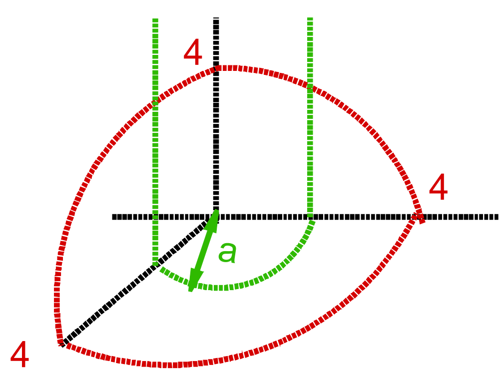
$$\int_0^{\pi/2}\frac{2}{3}\left(16-r^2\right)^{3/2}\Big]_{4cos\,\theta}^{0}d\theta=$$

$$\frac{2}{3}\int_0^{\pi/2}\left(16^{3/2}-[16-16cos^2\theta]^{3/2}\right)d\theta=$$

$$\frac{2}{3}\int_0^{\pi/2}\left(64-64sin^3\theta\right)d\theta=\frac{128}{3}\int_0^{\pi/2}\left(1-sin^3\theta\right)d\theta=$$

$$38.576\,.$$

33. Find a such that the volume inside the hemisphere $z=\sqrt{16-x^2-y^2}$ and outside the cylinder $x^2+y^2=a^2$ is $1/2$ the volume of the hemisphere.



$$\frac{1}{2}\text{ volume hemisphere} = \frac{1}{2}\cdot\frac{4}{3}\pi r^3 = \frac{2}{3}\pi\left(4^3\right) = \frac{128\pi}{3}\,.$$

The volume inside the hemisphere and outside the cylinder is

$$8\int_0^{\pi/2}\int_a^4 \sqrt{16-r^2}\, r\,dr\,d\theta = -4\int_0^{\pi/2}\int_{r=a}^{r=4} u^{1/2}du d\theta$$

where $u = 16 - r^2 \;\Rightarrow\; du = -2rdr$

$$= -4\int_0^{\pi/2} \frac{u^{3/2}}{3/2}\Bigg]_{r=a}^{r=4} d\theta = -\frac{8}{3}\int_0^{\pi/2}\left(16-r^2\right)^{3/2}\Bigg]_a^4 d\theta =$$

$$\frac{8}{3}\int_0^{\pi/2}\left(16-r^2\right)^{3/2}\Bigg]_4^a d\theta = \frac{8}{3}\int_0^{\pi/2}\left(16-a^2\right)^{3/2} d\theta =$$

$$\frac{8}{3}\left(16-a^2\right)^{3/2}\theta\Bigg]_0^{\pi/2} = \frac{8}{3}\cdot\frac{\pi}{2}\left(16-a^2\right)^{3/2} = \frac{4\pi}{3}\left(16-a^2\right)^{3/2}.$$

If this volume is $\frac{1}{2}$ volume of the hemisphere then

$$\frac{4}{3}\pi\left(16-a^2\right)^{3/2} = \frac{128\pi}{3} \;\Rightarrow\; 4\left(16-a^2\right)^{3/2} = 128$$

$$\Rightarrow\; \left(16-a^2\right)^{3/2} = 32 \;\Rightarrow\; 16-a^2 = 32^{2/3}$$

$$\Rightarrow\; a^2 = 16 - \left(\sqrt[3]{32}\right)^2 \;\Rightarrow\; a^2 = 16 - \left(2\sqrt[3]{4}\right)^2 \;\Rightarrow$$

$$a = \sqrt{16 - 4\sqrt[3]{16}} \;\Rightarrow\; a = 2\sqrt{4 - \sqrt[3]{16}}$$

$$\Rightarrow\; a = 2\sqrt{4 - 2\sqrt[3]{2}}\,.$$