MATH 260
SUMMER 2022
EXAM 1

NAME:

1. [10 pts.] Determine the values of g and h such that the system

$$\begin{cases} x_1 + 3x_2 = g \\ 4x_1 + hx_2 = 8 \end{cases}$$

has (a) no solution; (b) a unique solution; and (c) many solutions.

2. [10 pts.] Find the general solution to the system having augmented matrix

$$\begin{bmatrix} 1 & 4 & 0 & 7 \\ 2 & 7 & 0 & 10 \end{bmatrix}$$

3. [10 pts.] Determine if $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$:

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 5 \\ 5 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 2 \\ 0 \\ 8 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -5 \\ 11 \\ -7 \end{bmatrix}.$$

4. [10 pts.] Write the augmented matrix for the linear system that corresponds to the matrix equation $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ -3 & -1 & 2 \\ 0 & 5 & 3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

then solve the system and write the solution as a vector.

5. [10 pts.] Given

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 4 \\ -1 \\ -5 \end{bmatrix},$$

prove or disprove that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

6. [10 pts.] When solutions of sodium phosphate and barium nitrate are mixed, the result is barium phosphate and sodium nitrate. The unbalanced equation is

$$Na_3PO_4 + Ba(NO_3)_2 \longrightarrow Ba_3(PO_4)_2 + NaNO_3.$$

Use the vector equation approach to balance the equation.

7. [10 pts.] Find the values of h for which the vectors are linearly dependent:

$$\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} -5 \\ 7 \\ 8 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ h \end{bmatrix}.$$

8. [10 pts.] Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\mathbf{x}$ into $x_1\mathbf{v}_1 + x_2\mathbf{v}_2$. Find a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x}$.

9. [10 pts.] An **affine transformation** $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, with $A \in \mathbb{R}^{m\times n}$ (i.e. A is an $m \times n$ matrix) and $\mathbf{b} \in \mathbb{R}^m$. Show that T is not a linear transformation when $\mathbf{b} \neq \mathbf{0}$.

10. [10 pts. each] Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation such that $T(\mathbf{e}_1) = (1, 3)$, $T(\mathbf{e}_2) = (4, -7)$, and $T(\mathbf{e}_3) = (-5, 4)$.

   (a) Prove or disprove that T one-to-one.

   (b) Prove or disprove that T onto.

11. [10 pts.] Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation such that $T(\mathbf{x}) = B\mathbf{x}$ for some $m \times n$ matrix B. Show that if A is the standard matrix for T, then $A = B$.

12. [10 pts.] Let

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}.$$

Find the values of k, if any, for which $AB = BA$.

Theorem 1: Suppose equation $A\mathbf{x} = \mathbf{b}$ is consistent for some vector $\mathbf{b}$, and let $\mathbf{p}$ be a particular solution. If S and S_h are the solution sets for $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$, respectively, then $S = \{\mathbf{p}+\mathbf{v}_h : \mathbf{v}_h \in S_h\}$.

Theorem 2: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. Then:

a) T is onto if and only if the columns of A span $\mathbb{R}^m$.

b) T is one-to-one if and only if the columns of A are linearly independent.