MATH 260
SUMMER 2021
EXAM 3

NAME:

1. Let

$$A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -3 \end{bmatrix}.$$

   (a) [10 pts.] Find the eigenvalues of A either directly or using the characteristic equation.

   (b) [10 pts.] Find a basis just for the eigenspace corresponding to the *largest* eigenvalue of A.

2. [10 pts.] Show that if A^2 equals the zero matrix, then A has only 0 for an eigenvalue.

3. [15 pts.] Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix};$$

   that is, find invertible matrix P and diagonal matrix D such that $A = PDP^{-1}$.

4. [10 pts.] Find the $\mathcal{B}$-matrix for the transformation $\mathbf{x} \mapsto A\mathbf{x}$ with $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, where

$$A = \begin{bmatrix} -1 & 4 \\ -2 & 3 \end{bmatrix}, \quad \mathbf{b}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

5. [10 pts.] Find a unit vector in the direction of the vector $\mathbf{v} = \begin{bmatrix} -6 \\ 4 \\ -3 \end{bmatrix}$.

6. [10 pts.] Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} 8 \\ -4 \\ -3 \end{bmatrix}.$$

   Express $\mathbf{x}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ using the fact that the set $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal.

7. [10 pts.] Prove that if U is an $m \times n$ matrix with orthonormal columns, then $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. (Hint: there's a theorem that says something about $U^\top U$.)

8. [10 pts.] Find the best approximation to $\mathbf{x}$ by vectors of the form $c_1\mathbf{u}_1 + c_2\mathbf{u}_2$, where

$$\mathbf{x} = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 = \begin{bmatrix} 2 \\ 0 \\ -1 \\ -3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix}.$$

9. 10 pts. Find an orthogonal basis for Col A, where

$$A = \begin{bmatrix} 3 & -8 & 3 \\ 1 & -4 & -3 \\ -1 & 6 & 6 \\ 1 & -2 & 6 \end{bmatrix}.$$

10. 10 pts. Suppose $\mathbb{P}_3$ has inner product given by

$$\langle p, q \rangle = p(-3)q(-3) + p(-1)q(-1) + p(1)q(1) + p(3)q(3).$$

Define polynomials $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = \frac{1}{4}t^2 - \frac{5}{4}$. Determine the best approximation to $r(t) = t^3$ by polynomials in Span$\{p_0, p_1, p_2\}$.

11. 10 pts. Recall $\mathcal{C}[a, b]$ denotes the vector space of real-valued functions that are continuous on the interval $[a, b]$. For $f, g \in \mathcal{C}[a, b]$ define

$$\langle f, g \rangle = \int_a^b f(t)g(t)\,dt,$$

which is an inner product on $\mathcal{C}[a, b]$. Find an orthogonal basis for the subspace of $\mathcal{C}[-3, 3]$ that is spanned by polynomials 3, $2t$, t^2.