

1. 10 pts. An $n \times n$ symmetric matrix \mathbf{A} with real entries is called positive definite if $\mathbf{x}^\top \mathbf{A} \mathbf{x} > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \neq \mathbf{0}$. Prove or disprove that

$$\mathbf{A} = \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$$

is positive definite.

2. 15 pts. Find an orthonormal basis for the subspace of \mathbb{R}^3 generated by the vectors $(2, 1, 1)$ and $(1, 3, -1)$.

3. 15 pts. Let V be the vector space of all continuous functions $f : [0, 1] \rightarrow \mathbb{R}$, and define a scalar product on V by

$$\langle f, g \rangle = \int_0^1 f(t)g(t) dt$$

for all $f, g \in V$. Let U be the subspace of V generated by $\{1, t, t^2\}$; that is, $U = \text{Span}\{1, t, t^2\}$. Find an orthonormal basis for U .

4. 10 pts. Evaluate the determinant:

$$\begin{vmatrix} 4 & -2 & 3b \\ 0 & a & -13 \\ 0 & 0 & a \end{vmatrix}.$$

5. 10 pts. Evaluate the determinant, which will be a polynomial in t :

$$\begin{vmatrix} -2 & 2 & 3 & -4 \\ 1 & 1 & 6 & 3 \\ -1 & 0 & 1 & -1 \\ -2 & t & 6 & -5 \end{vmatrix}.$$

6. 10 pts. Solve the system using Cramer's Rule.

$$\begin{cases} 3x + y - z = 0 \\ x + y + z = 0 \\ y - z = 1 \end{cases}$$

7. 10 pts. each Let

$$\mathbf{B} = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}.$$

- (a) Find the characteristic equation for \mathbf{B} , and use it to find the eigenvalues of \mathbf{B} .
(b) Find a basis for the eigenspace corresponding to each real eigenvalue.

8. 6 pts. each The matrix

$$\mathbf{N} = \begin{bmatrix} -2 & 8 \\ -1 & 4 \end{bmatrix}$$

is diagonalizable.

- (a) Find the characteristic equation for \mathbf{N} , and use it to find the eigenvalues of \mathbf{N} .
- (b) For each eigenvalue of \mathbf{N} find the basis for the corresponding eigenspace.
- (c) Find an invertible matrix \mathbf{P} and a diagonal matrix \mathbf{D} such that $\mathbf{N} = \mathbf{PDP}^{-1}$.
- (d) Find \mathbf{N}^{10} .
- (e) Find $\mathbf{N}^{1/2}$.