MATH 260
SUMMER 2017 NAME:
EXAM 2

1. $\boxed{\text{10 pts.}}$ Let V be a vector space. Given a set $S \subset V$, for $a \in \mathbb{R}$ and $\mathbf{b} \in V$ define

$$aS + \mathbf{b} = \{a\mathbf{v} + \mathbf{b} : \mathbf{v} \in S\}.$$

Prove that if S is convex, then so too is $aS + \mathbf{b}$.

2. $\boxed{\text{10 pts.}}$ Let V be a subspace of $\mathbb{R}^2$. What are the possible dimensions of V? Show that if $V \neq \mathbb{R}^2$, then either $V = \{\mathbf{0}\}$, or V is a line through the origin.

3. $\boxed{\text{10 pts.}}$ Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{B}$ an $n \times r$ matrix. Show that the columns of $\mathbf{AB}$ are a linear combination of the columns of $\mathbf{A}$, and go on to prove that $\operatorname{rank}(\mathbf{AB}) \leq \operatorname{rank}(\mathbf{A})$.

4. $\boxed{\text{10 pts.}}$ Find the rank of the matrix, justifying your answer:

$$\mathbf{M} = \begin{bmatrix} 1 & -2 & 0 & 4 \\ 3 & 1 & 1 & 0 \\ -1 & -5 & -1 & 8 \\ 3 & 8 & 2 & -12 \end{bmatrix}.$$

5. $\boxed{\text{10 pts.}}$ Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the mapping defined by $F(x, y) = (e^x \sin y, e^x \cos y)$. Describe the image under F of the line $x = c$, where c is a constant.

6. $\boxed{\text{10 pts.}}$ Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear mapping such that $L(1, 1) = (2, 1)$ and $L(-1, 1) = (6, 3)$. Find $L(1, 0)$.

7. $\boxed{\text{10 pts.}}$ Let V, W be vector spaces and $L : V \to W$ a linear mapping. Suppose $\mathbf{w}_1, \ldots, \mathbf{w}_n \in W$ are linearly independent and $\mathbf{v}_1, \ldots, \mathbf{v}_n \in V$ are such that $L(\mathbf{v}_k) = \mathbf{w}_k$ for $1 \leq k \leq n$. Show that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

8. $\boxed{\text{10 pts.}}$ Let $L : V \to W$ be a linear mapping. Assume $\dim V > \dim W$. Show that the kernel of L is not $\{\mathbf{0}\}$.

9. $\boxed{\text{10 pts.}}$ Find the dimension of the subspace of $\mathbb{R}^6$ perpendicular (i.e. orthogonal) to the two vectors $(1, 1, -2, 3, 4, 5)$ and $(1, 0, 0, 2, 0, 8)$.

10. $\boxed{\text{10 pts.}}$ Find the matrix corresponding to the linear mapping $L : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$L\big([x_1, x_2, x_3, x_4]^\top\big) = [x_3, 0, -2x_2]^\top$$

with respect to the standard bases.

11. $\boxed{\text{10 pts.}}$ Let $P : V \to V$ be a linear map such that $P \circ P = P$. Show that $V = \operatorname{Ker}(P) + \operatorname{Img}(P)$.

12. $\boxed{\text{10 pts.}}$ Let $L : \mathbb{R}^{1\times 2} \to \mathbb{R}^{1\times 2}$ be the linear transformation defined by

$$L(x, y) = [2x + y, 3x - 5y].$$

Show that L is invertible.

13. $\boxed{\text{10 pts.}}$ The ordered sets

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix} \right) \quad \text{and} \quad \mathcal{C} = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \end{bmatrix} \right)$$

are bases for $\mathbb{R}^2$. Find the change of basis matrix $\mathbf{I}_{\mathcal{BC}}$ (a.k.a. transition matrix) for changing from the basis $\mathcal{B}$ to the basis $\mathcal{C}$.

Vector Space Axioms:

VS1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for any $\mathbf{u}, \mathbf{v} \in V$

VS2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

VS3. There exists some $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for any $\mathbf{u} \in V$

VS4. For each $\mathbf{u} \in V$ there exists some $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

VS5. For any $a \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

VS6. For any $a, b \in \mathbb{F}$ and $\mathbf{u} \in V$, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

VS7. For any $a, b \in \mathbb{F}$ and $\mathbf{u} \in V$, $a(b\mathbf{u}) = (ab)\mathbf{u}$

VS8. For all $\mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$