Math 260
Summer 2016
Exam 2

Name:

1. $\boxed{\text{10 pts.}}$ Let $\mathbf{a} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ be fixed, with $\mathbf{a} \neq \mathbf{0}$. Prove that the set

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^\top \mathbf{a} \geq c\}$$

is convex.

2. $\boxed{\text{10 pts.}}$ Find a basis for the vector space of *strictly* lower-triangular 3×3 matrices with real entries. These are real 3×3 matrices with all entries on the main diagonal and above the main diagonal equal to zero. What is the dimension of the vector space?

3. $\boxed{\text{10 pts.}}$ Find the rank of the matrix, justifying your answer:

$$\mathbf{H} = \begin{bmatrix} -2 & 2 & 3 & -4 & -1 \\ 1 & 1 & -2 & 3 & 1 \\ -1 & 3 & 1 & -1 & 0 \\ -2 & 2 & 6 & -4 & 2 \end{bmatrix},$$

4. $\boxed{\text{10 pts.}}$ Prove or disprove that the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} xy \\ x \end{bmatrix}$$

is linear.

5. $\boxed{\text{10 pts.}}$ Let $\mathcal{P}$ denote the vector space of all polynomials in x with real coefficients, and let $D^n : \mathcal{P} \to \mathcal{P}$ denote the nth derivative operator. What is the kernel of D^2? In general, what is the kernel of D^n?

6. $\boxed{\text{10 pts.}}$ Find the dimension of the space of solutions of the system below. Also find a basis for the space of solutions.

$$\begin{cases} x + y + z = 0 \\ x - y = 0 \\ y + z = 0 \end{cases}$$

7. $\boxed{\text{10 pts.}}$ Find the matrix corresponding to the linear mapping $L : \mathbb{R}^4 \to \mathbb{R}^3$ given by

$$L\big([x_1, x_2, x_3, x_4]^\top\big) = [x_2, x_4, 0]^\top$$

with respect to the standard bases.

8. Let $P : V \to V$ be a linear map such that $P \circ P = P$.
   (a) $\boxed{\text{10 pts.}}$ Show that $V = \mathrm{Ker}(P) + \mathrm{Img}(P)$.
   (b) $\boxed{\text{5 pts.}}$ Show that $\mathrm{Ker}(P) \cap \mathrm{Img}(P) = \{\mathbf{0}\}$.

9. $\boxed{\text{10 pts.}}$ Let $L : \mathbb{R}^{1\times 2} \to \mathbb{R}^{1\times 2}$ be the linear transformation defined by

$$L(x, y) = [2x + y, 3x - 5y].$$

Show that L is invertible.

10. $\boxed{\text{10 pts.}}$ Let $L : V \to V$ be a linear transformation such that $L^2 + 2L + I = O$. Show that L is invertible.

Vector Space Axioms:

VS1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ for any $\mathbf{u}, \mathbf{v} \in V$

VS2. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$

VS3. There exists some $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for any $\mathbf{u} \in V$

VS4. For each $\mathbf{u} \in V$ there exists some $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

VS5. For any $a \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in V$, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$

VS6. For any $a, b \in \mathbb{F}$ and $\mathbf{u} \in V$, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$

VS7. For any $a, b \in \mathbb{F}$ and $\mathbf{u} \in V$, $a(b\mathbf{u}) = (ab)\mathbf{u}$

VS8. For all $\mathbf{u} \in V$, $1\mathbf{u} = \mathbf{u}$