MATH 260
SUMMER 2015 NAME:
EXAM 2

Vector Space Axioms:
VS1. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$ for any $\mathbf{u},\mathbf{v}\in V$
VS2. $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$ for any $\mathbf{u},\mathbf{v},\mathbf{w}\in V$
VS3. There exists some $\mathbf{0}\in V$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$ for any $\mathbf{u}\in V$
VS4. For each $\mathbf{u}\in V$ there exists some $-\mathbf{u}\in V$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$
VS5. For any $a\in\mathbb{F}$ and $\mathbf{u},\mathbf{v}\in V$, $a(\mathbf{u}+\mathbf{v})=a\mathbf{u}+a\mathbf{v}$
VS6. For any $a,b\in\mathbb{F}$ and $\mathbf{u}\in V$, $(a+b)\mathbf{u}=a\mathbf{u}+b\mathbf{u}$
VS7. For any $a,b\in\mathbb{F}$ and $\mathbf{u}\in V$, $a(b\mathbf{u})=(ab)\mathbf{u}$
VS8. For all $\mathbf{u}\in V$, $1\mathbf{u}=\mathbf{u}$

1. $\boxed{\text{10 pts.}}$ Let V be a vector space. Use the vector space axioms to prove the following cancellation rule: If $\mathbf{u},\mathbf{v},\mathbf{w}\in V$, then $\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$ implies that $\mathbf{u}=\mathbf{v}$.

2. $\boxed{\text{10 pts.}}$ Letting $\mathbf{x}$ denote $[x_1,x_2,x_3]^\top$, prove or disprove that

$$U=\{\mathbf{x}:\|\mathbf{x}\|=|x_1|+|x_2|\}$$

is a subspace of $\mathbb{R}^3$.

3. $\boxed{\text{10 pts.}}$ Write $\mathbf{v}$ as a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, if possible:

$$\mathbf{v}=\begin{bmatrix}7\\16\\-3\end{bmatrix},\quad \mathbf{u}_1=\begin{bmatrix}4\\2\\1\end{bmatrix},\quad \mathbf{u}_2=\begin{bmatrix}4\\-3\\2\end{bmatrix},\quad \mathbf{u}_3=\begin{bmatrix}1\\3\\-1\end{bmatrix}.$$

4. $\boxed{\text{10 pts. each}}$ Let

$$\mathbf{A}=\begin{bmatrix}1&2&1\\1&3&3\\0&2&4\end{bmatrix}.$$

(a) Find a basis for $\mathrm{Col}(\mathbf{A})$.

(b) Find a basis for $\mathrm{Nul}(\mathbf{A})$.

5. $\boxed{\text{10 pts.}}$ Show that for any two matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{AB}$ exists and $\mathbf{A}$ is invertible we have $\mathrm{Nul}(\mathbf{B})=\mathrm{Nul}(\mathbf{AB})$.

6. $\boxed{\text{10 pts.}}$ For

$$\mathbf{C}=\begin{bmatrix}1&1&1&2&-1\\0&2&0&0&4\end{bmatrix}$$

find the dimension of $\mathrm{Col}(\mathbf{C})$, $\mathrm{Row}(\mathbf{C})$ and $\mathrm{Nul}(\mathbf{C})$.

7. The ordered set

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$$

is a basis for $\mathbb{R}^2$. The standard basis is $\mathcal{E} = ([1,0]^\top, [0,1]^\top)$.

(a) $\boxed{\text{10 pts.}}$ Find the change of basis matrix $\mathbf{I}_{\mathcal{E}\mathcal{B}}$ (a.k.a. transition matrix) for changing from the basis $\mathcal{E}$ to the basis $\mathcal{B}$.

(b) $\boxed{\text{5 pts.}}$ Use $\mathbf{I}_{\mathcal{E}\mathcal{B}}$ to find the $\mathcal{B}$-coordinates of $\mathbf{x} = [2,-5]^\top$. (That is, find the components of $\mathbf{x}$ in the basis $\mathcal{B}$.)

8. Let $L : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear tranformation for which

$$L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

(a) $\boxed{\text{10 pts.}}$ Find the matrix $[L]$ (with respect to the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$) that represents the transformation L.

(b) $\boxed{\text{5 pts.}}$ Find a general formula for L. That is, find the 3×1 matrix $L([x,y]^\top)$.

(c) $\boxed{\text{10 pts.}}$ Find the range and kernel of L. Is L one-to-one?