Math 260
Summer 2013 Name:
Exam 3

1. [10 pts.] Use a linear transformation and appropriate theorems to determine the dimension of the subspace S of $\mathbb{R}^n$ given by

$$S = \{\langle a_1, \ldots, a_n \rangle : a_1 + \cdots + a_n = 0\}.$$

2. Let $\mathrm{Mat}_n(\mathbb{R})$ be the vector space consisting of all $n \times n$ matrices with real-valued entries, and define $Q : \mathrm{Mat}_n(\mathbb{R}) \to \mathrm{Mat}_n(\mathbb{R})$ by

$$Q(\mathbf{A}) = \frac{\mathbf{A} - \mathbf{A}^\top}{2},$$

where $\mathbf{A}^\top$ denotes the transpose of the matrix $\mathbf{A}$.

(a) [5 pts.] Show that Q is linear.

(b) [10 pts.] Describe the kernel of Q, and determine its dimension.

(c) [5 pts.] What is the image of Q?

3. [10 pts.] Let $L : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 - 7x_3 \\ 4x_1 - 6x_2 + 5x_3 \end{bmatrix}$$

What is the matrix corresponding to L?

4. [10 pts.] Two bases for the vector space $\mathbb{P}_2 = \{a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R}\}$ are

$$\mathcal{B} = \{1, x, x^2\} \quad \text{and} \quad \mathcal{D} = \{1, 1+x, 1+x+x^2\}.$$

Find the coordinate transformation (a.k.a. change of basis) matrix from $\mathcal{B}$ to $\mathcal{D}$, and also from $\mathcal{D}$ to $\mathcal{B}$.

5. [10 pts.] Find the dimension of the space of solutions to the system of equations

$$\begin{cases} 2x + y - z = 0 \\ 2x + y + z = 0 \end{cases}$$

and then find a basis for the space of solutions.

6. Let $P : V \to V$ be a linear map such that $P \circ P = P$.

(a) [10 pts.] Show that $V = \mathrm{Ker}(P) + \mathrm{Im}(P)$.

(b) [5 pts.] Show that $\mathrm{Ker}(P) \cap \mathrm{Im}(P) = \{\mathbf{0}\}$.

7. [10 pts.] Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$L(x, y) = (2x + y, 3x - 5y).$$

Show that L is invertible.

8. [10 pts.] Let $L : V \to V$ be a linear transformation such that $L^2 + 2L + I = O$. Show that L is invertible.

9. Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. Given column vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^\top \mathbf{A} \mathbf{y}.$$

   (a) [10 pts.] Show that this formula satisfies the first three axioms of an inner product (see below).

   (b) [5 pts.] Give an example of a nonzero symmetric 2×2 matrix $\mathbf{A}$ such that the fourth axiom of an inner product is not satisfied in general.

10. [15 pts.] Find an orthonormal basis for the subspace of $\mathbb{R}^4$ generated by $\langle 1, 1, 0, 0 \rangle$, $\langle 1, -1, 1, 1 \rangle$, and $\langle -1, 0, 2, 1 \rangle$.

Axioms of an Inner Product $\langle \ , \ \rangle$ for a vector space V over $\mathbb{R}$

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
3. $\langle x\mathbf{u}, \mathbf{v} \rangle = x\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, x\mathbf{v} \rangle$ for all $\mathbf{u} \in V$, $x \in \mathbb{R}$.
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u} \in V$, and $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ if $\mathbf{u} \neq \mathbf{0}$.