Math 260
Summer 2012 Name:
Exam 3

1. Let $P : V \to V$ be a linear map such that $P \circ P = P$.

   (a) [10 pts.] Show that $V = \mathrm{Ker}(P) + \mathrm{Im}(P)$.

   (b) [5 pts.] Show that $\mathrm{Ker}(P) \cap \mathrm{Im}(P) = \{\mathbf{0}\}$.

2. [10 pts.] Let U and W be subspaces of V. We call V the direct sum of U and W, written $V = U \oplus W$, if $V = U + W$ and $U \cap W = \{\mathbf{0}\}$.

   Let $V = U \oplus W$. Show that if $\mathbf{v} \in U + W$, then there exist *unique* vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ such that $\mathbf{v} = \mathbf{u} + \mathbf{w}$. That is, if $\mathbf{v} = \mathbf{u} + \mathbf{w}$ and $\mathbf{v} = \mathbf{u}' + \mathbf{w}'$ for $\mathbf{u}, \mathbf{u}' \in U$ and $\mathbf{w}, \mathbf{w}' \in W$, then $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$.

3. [10 pts.] Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by

$$L(x, y) = (2x + y, 3x - 5y).$$

   Show that L is invertible.

4. [10 pts.] Let $L : V \to V$ be a linear transformation such that $L^2 + 2L + I = O$. Show that L is invertible.

5. [10 pts.] Compute the determinant

$$\begin{vmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 3 & 1 & 5 & 7 \end{vmatrix}$$

   by row or column expansions (i.e. use the definition of determinant).

6. [10 pts.] Find the rank of the matrix

$$\begin{bmatrix} 3 & 5 & 1 & 4 \\ 2 & -1 & 1 & 1 \\ 8 & 9 & 3 & 9 \end{bmatrix}$$

   using determinants.

7. [10 pts.] Solve the system

$$\begin{cases} 2x - y + z = 1 \\ x + 3y - 2z = 0 \\ 4x - 3y + z = 2 \end{cases}$$

   using Cramer's Rule.

8. [10 pts.] Let $a \neq 0$. Prove that the eigenvectors of the matrix

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

generate a 1-dimensional vector space, and give a basis for the space.

9. Let

$$\mathbf{A} = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

(a) [5 pts.] Find the characteristic polynomial for $\mathbf{A}$.

(b) [5 pts.] Find the eigenvalues of $\mathbf{A}$.

(c) [10 pts.] Find bases for all eigenspaces of $\mathbf{A}$.

10. Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. Given column vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, define

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T \mathbf{A} \mathbf{y}.$$

(a) [10 pts.] Show that this formula satisfies the first three properties of an inner product (see below).

(b) [5 pts.] Give an example of a symmetric 2×2 matrix $\mathbf{A}$ such that the fourth property of an inner product is not satisfied in general.

Properties of an Inner Product $\langle \ , \ \rangle$

1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$.
2. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$.
3. $\langle x\mathbf{u}, \mathbf{v} \rangle = x\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, x\mathbf{v} \rangle$ for all $\mathbf{u} \in V$, $x \in \mathbb{R}$.
4. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u} \in V$, and $\langle \mathbf{u}, \mathbf{u} \rangle > 0$ if $\mathbf{u} \neq \mathbf{0}$.