MATH 260
SUMMER 2012 NAME:
EXAM 2

1. [10 pts.] Let $\mathbf{a} \in \mathbb{R}^n$ be a nonzero vector and let $c \in \mathbb{R}$ be fixed. Show that the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{a} \cdot \mathbf{x} \geq c$ is convex.

2. [10 pts.] Show that the vectors $\langle 1, 2\rangle$ and $\langle 1, 5\rangle$ are linearly independent.

3. [10 pts.] Let $\mathbf{v}_1, \ldots, \mathbf{v}_r \in \mathbb{R}^n$ be mutually orthogonal nonzero vectors. That is, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$, and $\mathbf{v}_i \neq \mathbf{0}$ for all $1 \leq i \leq r$. Prove that they are linearly independent.

4. [10 pts.] Find the rank of

$$\begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 3 \\ 4 & 8 & -12 \\ 1 & -1 & 5 \end{bmatrix}$$

by putting the matrix in either row echelon or column echelon form.

5. [10 pts.] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation defined by $T(x, y) = (3x, 7y)$. Describe the image under T of the points lying on the circle $x^2 + y^2 = 1$.

6. [10 pts.] Is the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x, y, z) = \langle 2x, y - z\rangle$ linear? If it is, prove it. If it isn't, give an example in which linearity is violated.

7. [10 pts.] Show that the image of a convex set under a linear transformation is convex.

8. [5 pts. each] Let $L : V \to W$ be a linear transformation, and $\dim(V) = \dim(W)$.

   (a) Show that if $\mathrm{Ker}(L) = \{\mathbf{0}\}$, then $\mathrm{Im}(L) = W$.

   (b) Show that if $\mathrm{Im}(L) = W$, the $\mathrm{Ker}(L) = \{\mathbf{0}\}$.

9. [10 pts.] Let $L : V \to W$ be a linear map. Show that if $\dim(V) > \dim(W)$, then $\mathrm{Ker}(L) \neq \{\mathbf{0}\}$.

10. [10 pts.] Find the dimension of the space of solutions to the system of equations

$$\begin{cases} 2x + y - z = 0 \\ 2x + y + z = 0 \end{cases}$$

and then find a basis for the space of solutions.

11. 10 pts. Find the dimension of the subspace of $\mathbb{R}^7$ that is orthogonal (i.e. perpendicular) to the vectors

$$\langle 1, 1, -2, 3, 4, 5, 6 \rangle \quad \text{and} \quad \langle 0, 0, 2, 1, 0, 7, 0 \rangle,$$

stating the reason for your answer.

12. 10 pts. Let $L : V \to W$ be a linear transformation. Show that if S is an arbitrary line in V, then $L(S)$—the image of S under L—is either a point or a line in W.

13. 10 pts. Let U be a subspace of $\mathbb{R}^n$. Prove that $U^\perp$ is also a subspace.

A few theorems (not comprehensive)

1. Let V be a vector space, and let $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} = V$. If $\mathbf{w}_1, \ldots, \mathbf{w}_n \in V$ for $n > m$, then $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly dependent.
2. If W is a subspace of V, then $\dim(W) \leq \dim(V)$.
3. If $L : V \to W$ is linear, then $\dim(V) = \dim(\text{Ker}(L)) + \dim(\text{Im}(L))$.