Math 260 Exam #3 Key (Summer 2022)

1 $\mathbf{0} \in \mathsf{U}$ since U is a subspace, and $T(\mathbf{0}) = \mathbf{0}$ since T is linear. Therefore $\mathbf{0} \in T(\mathsf{U})$.

Let $\mathbf{w}_1, \mathbf{w}_2 \in T(\mathsf{U})$, so there exists $\mathbf{u}_1, \mathbf{u}_2 \in \mathsf{U}$ such that $T(\mathbf{u}_k) = \mathbf{w}_k$. Because U is a subspace, $\mathbf{u}_1 + \mathbf{u}_2 \in \mathsf{U}$; and because T is linear, $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2) = \mathbf{w}_1 + \mathbf{w}_2$. Thus $\mathbf{w}_1 + \mathbf{w}_2 \in T(\mathsf{U})$.

Let $\mathbf{w} \in T(\mathsf{U})$ and $c \in \mathbb{R}$. There exists $\mathbf{u} \in \mathsf{U}$ such that $T(\mathbf{u}) = \mathbf{w}$, with $c\mathbf{u} \in \mathsf{U}$ also. Now, $T(c\mathbf{u}) = cT(\mathbf{u}) = c\mathbf{w}$ shows that $c\mathbf{w} \in T(\mathsf{U})$.

2 Let A be the given matrix. By definition $\operatorname{Nul} A = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$, so the job is to find a basis for the solution set to the homogeneous system with augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 2 & -8 & 1 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right].$$

We find that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5x_3 - 7x_5 \\ 4x_3 - 6x_5 \\ x_3 \\ 3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix},$$

where x_3 and x_5 are free, and so a basis for $\operatorname{Nul} A$ is the set

$$\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

3 Define the matrix $M = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5]$, so the space in question is $\operatorname{Col} M$. With a series of row operations get a new matrix B that is in echelon form:

$$M = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

The last two columns of B can each be expressed as linear combinations of the first three columns, so can be discarded. The same relationship must hold for the columns of M, so any basis for $\operatorname{Col} M$ must involve only $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ (i.e. the first three columns of M). But the first three columns of M must be linearly independent since the first three columns of B are. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set that spans $\operatorname{Col} M$, and therefore forms a basis for the space in question.

4 Form the matrix $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$. Then A is invertible since $\det A = 7 \neq 0$, and then by the Invertible Matrix Theorem the columns of A (i.e. the vectors in $\mathcal{B}$) form a basis for $\mathbb{R}^2$. As for the change-of-coordinates matrix, by inspection we have $P_{\mathcal{B}} = A$.

5 Let $\mathcal{B}$ be a basis for U, so $\mathcal{B}$ is a linearly independent set of n vectors in U such that $\operatorname{Span}\mathcal{B} = \mathsf{U}$. But $\mathsf{U} \subseteq \mathsf{V}$, so $\mathcal{B}$ is also a linearly independent set of n vectors in V, and because $\dim \mathsf{V} = n$, the Basis Theorem (Theorem 4.13) implies that $\mathcal{B}$ is a basis for V as well. Therefore $\mathsf{V} = \operatorname{Span}\mathcal{B} = \mathsf{U}$.

6 First we get $P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[\, [\mathbf{b}_1]_{\mathcal{C}} \ \ [\mathbf{b}_2]_{\mathcal{C}} \,\big]$. To find $[\mathbf{b}_1]_{\mathcal{C}}$ we find scalars x_1, x_2 such that $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 = \mathbf{b}_1$, which is a system with augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & 8 \end{bmatrix}.$$

Solving gives $x_1 = 3$ and $x_2 = -4$. To find $[\mathbf{b}_2]_{\mathcal{C}}$ we find y_1, y_2 such that $y_1\mathbf{c}_1 + y_2\mathbf{c}_2 = \mathbf{b}_2$, which is a system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & -5 \end{bmatrix}.$$

Solving gives $y_1 = -2$ and $y_2 = 3$. Therefore

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[\, [\mathbf{b}_1]_{\mathcal{C}} \ \ [\mathbf{b}_2]_{\mathcal{C}} \,\big] = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

7 For $(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3) = (1-2t+t^2, 3-5t+4t^2, 2t+3t^2)$ let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. Without calculation we have

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[\, [\mathbf{b}_1]_{\mathcal{C}} \ \ [\mathbf{b}_2]_{\mathcal{C}} \ \ [\mathbf{b}_3]_{\mathcal{C}} \,\big] = \begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix}.$$

The $\mathcal{C}$-coordinates of $t - t^2$ are $[t-t^2]_{\mathcal{C}} = [0 \ \ 1 \ \ -1]^\top$, and so the $\mathcal{B}$-coordinates are

$$[t-t^2]_{\mathcal{B}} = P_{\mathcal{B}\leftarrow\mathcal{C}}[t-t^2]_{\mathcal{C}} = (P_{\mathcal{C}\leftarrow\mathcal{B}})^{-1}[t-t^2]_{\mathcal{C}} = \begin{bmatrix} -23 & -9 & 6 \\ 8 & 3 & -2 \\ -3 & -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -15 \\ 5 \\ -2 \end{bmatrix}.$$

8 Suppose λ is an eigenvalue of A, so $\det(A - \lambda I) = 0$. But $\det M = \det M^\top$ for any square matrix M, so

$$\det(A^\top - \lambda I) = \det[A^\top - (\lambda I)^\top] = \det[(A-\lambda I)^\top] = \det(A - \lambda I) = 0,$$

and hence λ is an eigenvalue of $A^\top$. Now, if λ is an eigenvalue of $A^\top$ then it is an eigenvalue of $(A^\top)^\top = A$.

9 Characteristic polynomial is

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & -2 \\ 2 & 3-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 25.$$

Eigenvalues are the roots of $\det(A - \lambda I) = 0$, or $\lambda = 5$.

10 Bases for the eigenspaces $E_A(3)$ and $E_A(4)$ are

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \right\},$$

respectively. Therefore $A = PDP^{-1}$ for

$$P = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$