Math 260 Exam #2 Key (Summer 2022)

1 Using the definition of a matrix product I've used in class (could also use the less conventional definition on page 101 of the textbook), the ij-entry of AI_n is

$$[AI_n]_{ij} = \sum_{k=1}^{n} [A]_{ik}[I_n]_{kj} = [A]_{ij}[I_n]_{jj} = [A]_{ij},$$

since $[I_n]_{kj} = 0$ when $k \neq j$ and $[I_n]_{kj} = 1$ when $k = j$. So the ij-entry of AI_n equals the ij-entry of A, and we conclude that $AI_n = A$.

2 In the course of attempting to find A^{-1} we perform the row operations $-4\mathsf{R}_1 + \mathsf{R}_2$, $2\mathsf{R}_1 + \mathsf{R}_3$, and $-2\mathsf{R}_2 + \mathsf{R}_3$ on $[A \,|\, I]$ to obtain $[B \,|\, C]$ with

$$B = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(what C is doesn't matter). Since B only has two pivots and A is row equivalent to B, we conclude that A has only two pivot positions, and therefore A is not invertible by the invertible matrix theorem.

3 The standard matrix for T is

$$A = [T(\mathbf{e}_1)\ \ T(\mathbf{e}_2)] = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix},$$

and since $\det A = 2 \neq 0$, Theorem 3.4 implies that A is invertible, and so T is invertible by Theorem 2.9, with $T^{-1}(\mathbf{x}) = A^{-1}\mathbf{x}$. Now,

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 7 & 8 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix},$$

so that

$$T^{-1}(x_1, x_2) = \begin{bmatrix} \frac{7}{2} & 4 \\ \frac{5}{2} & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{7}{2}x_1 + 4x_2 \\ \frac{5}{2}x_1 + 3x_2 \end{bmatrix}.$$

4 The equation becomes

$$\begin{vmatrix} a+1 & b \\ c & d+1 \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad \Rightarrow \quad (a+1)(d+1) - bc = ad - bc + 1,$$

which simplifies to $a + d = 0$, and therefore the only needed condition is $d = -a$.

5 Since $\det I = 1$, Theorem 3.6 implies

$$1 = \det I = \det(C^\top C) = (\det C^\top)(\det C);$$

but $\det C^\top = \det C$ by Theorem 3.5, so $(\det C)^2 = 1$ and hence $|\det C| = 1$.

6 The system is $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 9 \\ -4 \end{bmatrix}.$$

We have

$$\det A = -1, \quad \det A_1(\mathbf{b}) = \begin{vmatrix} 9 & 2 \\ -4 & -1 \end{vmatrix} = -1, \quad \det A_2(\mathbf{b}) = \begin{vmatrix} -5 & 9 \\ 3 & -4 \end{vmatrix} = -7.$$

By Cramer's rule, $x_1 = \det A_1(\mathbf{b})/\det A = 1$ and $x_2 = \det A_1(\mathbf{b})/\det A = 7$.

7 We're given $\mathsf{H} = \{X \in \mathbb{R}^{2\times 4} : CX = O\}$. Since $O \in \mathbb{R}^{2\times 4}$ is such that $CO = O$, we have $O \in \mathsf{H}$. Also, for any $X, Y \in \mathsf{H}$ we have $CX = O$ and $CY = O$, and so by Theorem 2.2(b)

$$C(X + Y) = CX + CY = O + O = O,$$

so that $X + Y \in \mathsf{H}$. Finally, for $a \in \mathbb{R}$ and $X \in \mathsf{H}$, Theorem 2.2(d) yields $C(aX) = a(CX) = aO = O$, and thus $aX \in \mathsf{H}$. Therefore H is a subspace.

8 The zero function on $[a, b]$, which we'll denote here by 0, is certainly a continuous real-valued function, and since $0(a) = 0 = 0(b)$ it follows that $0 \in \mathsf{H}$.

Next, suppose $f, g \in \mathsf{H}$, so f and g are continuous and real-valued on $[a, b]$ such that $f(a) = f(b)$ and $g(a) = g(b)$. Adding the two equations gives $f(a) + g(a) = f(b) + g(b)$, and thus $(f + g)(a) = (f + g)(b)$, and since calculus informs us that $f + g$ must also be continuous and real-valued on $[a, b]$, we conclude that $f + g \in \mathsf{H}$. Therefore H is closed under vector addition.

Finally, for any $f \in \mathsf{H}$ and scalar $c \in \mathbb{R}$ we know that cf must be continuous and real-valued, and since $f(a) = f(b) \Rightarrow cf(a) = cf(b) \Rightarrow (cf)(a) = (cf)(b)$, we conclude that $cf \in \mathsf{H}$. Therefore H is closed under scalar multiplication. This shows that H is a subspace of $\mathcal{C}[a, b]$.

9 First, $\mathbf{0} = \mathbf{0} + \mathbf{0}$ for $\mathbf{0} \in \mathsf{H}, \mathsf{K}$, so that $\mathbf{0} \in \mathsf{H} + \mathsf{K}$ and $\mathsf{H} + \mathsf{K} \neq \varnothing$.

Let $\mathbf{x}, \mathbf{y} \in \mathsf{H} + \mathsf{K}$, so $\mathbf{x} = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{y} = \mathbf{u}_2 + \mathbf{v}_2$ for some $\mathbf{u}_1, \mathbf{u}_2 \in \mathsf{H}$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathsf{K}$. Now, $\mathbf{u}_1 + \mathbf{u}_2 \in \mathsf{H}$ and $\mathbf{v}_1 + \mathbf{v}_2 \in \mathsf{K}$ since H and K are vector spaces, and because $\mathbf{x} + \mathbf{y} = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$ it follows that $\mathbf{x} + \mathbf{y} \in \mathsf{H} + \mathsf{K}$ also. Thus $\mathsf{H} + \mathsf{K}$ is closed under vector addition.

Now let $\mathbf{w} \in \mathsf{H}+\mathsf{K}$ and $c \in \mathbb{R}$. There exists $\mathbf{u} \in \mathsf{H}$ and $\mathbf{v} \in \mathsf{K}$ such that $\mathbf{w} = \mathbf{u}+\mathbf{v}$, and since H and K are vector spaces we also have $c\mathbf{u} \in \mathsf{H}$ and $c\mathbf{v} \in \mathsf{K}$, so that $c\mathbf{w} = c(\mathbf{u}+\mathbf{v}) = c\mathbf{u}+c\mathbf{v} \in \mathsf{H}+\mathsf{K}$. Therefore $\mathsf{H} + \mathsf{K}$ is closed under scalar multiplication.