MATH 260 EXAM #3 KEY (SUMMER 2021)

1a A is a triangular matrix, and so the eigenvalues are immediately seen to be -3, 0, 4.

1b We find a basis for $E_A(4)$, the eigenspace of A corresponding to the largest eigenvalue 4. We have

$$E_A(4) = \{\mathbf{x} : A\mathbf{x} = 4\mathbf{x}\} = \{\mathbf{x} : (A - 4I)\mathbf{x} = \mathbf{0}\},$$

so the job is simply to solve the system $(A - 4I)\mathbf{x} = \mathbf{0}$ to get $\text{Span}\left\{\begin{bmatrix} 7 & 0 & 1 \end{bmatrix}^\top\right\}$, and so $\left\{\begin{bmatrix} 7 & 0 & 1 \end{bmatrix}^\top\right\}$ is a basis.

2 We have $A^2 = O$. Suppose $A\mathbf{x} = \lambda\mathbf{x}$ for some $\mathbf{x} \neq \mathbf{0}$. Then

$$\mathbf{0} = O\mathbf{x} = A^2\mathbf{x} = A(A\mathbf{x}) = A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x},$$

which implies $\lambda^2 = 0$, and hence $\lambda = 0$. Therefore we conclude that 0 is the only eigenvalue of A.

3 The characteristic equation $\det(A - \lambda I) = 0$ becomes $(\lambda - 2)(\lambda - 1) = 12$, which has solutions -2, 5. Since

$$E_A(-2) = \{\mathbf{x} : A\mathbf{x} = -2\mathbf{x}\} = \{\mathbf{x} : (A + 2I)\mathbf{x} = \mathbf{0}\} = \text{Span}\left\{\begin{bmatrix} -3 \\ 4 \end{bmatrix}\right\}$$

and

$$E_A(5) = \{\mathbf{x} : A\mathbf{x} = 5\mathbf{x}\} = \{\mathbf{x} : (A - 5I)\mathbf{x} = \mathbf{0}\} = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\},$$

we let

$$D = \begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}.$$

4 For $T(\mathbf{x}) = A\mathbf{x}$ we find $[T]_{\mathcal{B}} = \big[[T(\mathbf{b}_1)]_{\mathcal{B}} \;\; [T(\mathbf{b}_2)]_{\mathcal{B}}\big]$, where

$$[T(\mathbf{b}_1)]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad [T(\mathbf{b}_2)]_{\mathcal{B}} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

with x_1, x_2, y_1, y_2 such that $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = T(\mathbf{b}_1) = A\mathbf{b}_1$ and $y_1\mathbf{b}_1 + y_2\mathbf{b}_2 = T(\mathbf{b}_2) = A\mathbf{b}_2$. These are two systems of equations having the same coefficient matrix, and we solve both simultaneously:

$$\left[\begin{array}{cc|cc} 3 & -1 & 5 & 5 \\ 2 & 1 & 0 & 5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 2 & 1 & 0 & 5 \\ 3 & -1 & 5 & 5 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 1 & 2 \\ 0 & 1 & -2 & 1 \end{array}\right].$$

From this we obtain

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}.$$

5 $\mathbf{v}/\|\mathbf{v}\| = \dfrac{\mathbf{v}}{\sqrt{61}} = \begin{bmatrix} -6/\sqrt{61} \\ 4/\sqrt{61} \\ -3/\sqrt{61} \end{bmatrix}.$

6 $\mathbf{x} = \sum_{k=1}^{3} \left(\frac{\mathbf{x} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k} \right) \mathbf{u}_k = \frac{5}{2}\mathbf{u}_1 - \frac{3}{2}\mathbf{u}_2 + 2\mathbf{u}_3.$

7 By a theorem we have $U^\top U = I$, and so, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$(U\mathbf{x}) \cdot (U\mathbf{y}) = (U\mathbf{x})^\top (U\mathbf{y}) = \mathbf{x}^\top U^\top U\mathbf{y} = \mathbf{x}^\top I\mathbf{y} = \mathbf{x}^\top \mathbf{y} = \mathbf{x} \cdot \mathbf{y}.$$

8 Let $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. The best approximation is

$$\hat{\mathbf{x}} = \text{proj}_W \mathbf{x} = \sum_{k=1}^{2} \left(\frac{\mathbf{x} \cdot \mathbf{u}_k}{\mathbf{u}_k \cdot \mathbf{u}_k} \right) \mathbf{u}_k = \frac{1}{14} \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} + \frac{4}{49} \begin{bmatrix} 5 \\ -2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 27/49 \\ -8/49 \\ 25/98 \\ -5/98 \end{bmatrix}.$$

9 Let

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -8 \\ -4 \\ 6 \\ -2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ -3 \\ 6 \\ 6 \end{bmatrix}.$$

Set $\mathbf{w}_1 = \mathbf{u}_1$. By the Gram-Schmidt procedure,

$$\mathbf{w}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix},$$

and

$$\mathbf{w}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \right) \mathbf{w}_1 - \left(\frac{\mathbf{u}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \right) \mathbf{w}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}.$$

Orthogonal basis for $\text{Col}\,\mathbf{A}$ is $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

10 Best approximation is

$$\hat{p} = \frac{\langle r, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 + \frac{\langle r, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1 + \frac{\langle r, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2.$$

Now,

$$\langle r, p_0 \rangle = r(-3)p_0(-3) + r(-1)p_0(-1) + r(1)p_0(1) + r(3)p_0(3) = 0,$$

and similarly $\langle r, p_1 \rangle = 164$, $\langle r, p_2 \rangle = 0$, $\langle p_1, p_1 \rangle = 20$. Note we don't need $\langle p_2, p_2 \rangle$. We finally obtain

$$\hat{p}(t) = \frac{164}{20} p_1(t) = \frac{41}{5} t.$$

11 Let $\mathbf{u}_1 = 3$, $\mathbf{u}_2 = 2t$, $\mathbf{u}_3 = t^2$. Set $\mathbf{w}_1 = \mathbf{u}_1 = 3$. By the Gram-Schmidt procedure,

$$\mathbf{w}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 = 2t - \frac{\int_{-3}^{3} 6t\,dt}{\int_{-3}^{3} 9\,dt}(3) = 2t,$$

and

$$\mathbf{w}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle}\mathbf{w}_1 - \frac{\langle \mathbf{u}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle}\mathbf{w}_2 = t^2 - \frac{\int_{-3}^{3} 3t^2\,dt}{\int_{-3}^{3} 9\,dt}(3) - \frac{\int_{-3}^{3} 2t^3\,dt}{\int_{-3}^{3} 4t^2\,dt}(2t) = t^2 - 3.$$

$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis.