Math 260 Exam #2 Key (Summer 2021)

1 The equation becomes

$$\begin{vmatrix} a+1 & b \\ c & d+1 \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \quad\Rightarrow\quad (a+1)(d+1) - bc = ad - bc + 1,$$

which simplifies to $a + d = 0$, and therefore the only needed condition is $d = -a$.

2 The system is $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -3 & 0 & 2 \\ 0 & 1 & -2 \end{bmatrix} \quad\text{and}\quad \mathbf{b} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix},$$

so that $\det A = -8$,

$$\det A_1(\mathbf{b}) = \begin{vmatrix} 3 & 1 & 0 \\ 0 & 0 & 2 \\ 2 & 1 & -2 \end{vmatrix} = -2, \quad \det A_2(\mathbf{b}) = \begin{vmatrix} 1 & 3 & 0 \\ -3 & 0 & 2 \\ 0 & 2 & -2 \end{vmatrix} = -22,$$

and

$$\det A_3(\mathbf{b}) = \begin{vmatrix} 1 & 1 & 3 \\ -3 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -3.$$

Cramer's Rule then gives $(x_1, x_2, x_3) = (\frac{1}{4}, \frac{11}{4}, \frac{3}{8})$.

3 Here we find that

$$W = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix} \right\},$$

so W is a subspace of $\mathbb{R}^3$ since it is a nonempty subset of $\mathbb{R}^3$ that is the span of a set of vectors in $\mathbb{R}^3$. (By Theorem 4.1 the span of any set of vectors in a vector space V is a subspace of V.)

4 Let W denote the set. The zero function on $[a, b]$, which we'll denote here by $\mathbf{0}$, is certainly a continuous real-valued function, and since $\mathbf{0}(a) = 0 = \mathbf{0}(b)$ it follows that $\mathbf{0} \in W$.

Next, suppose $f, g \in W$, so f and g are continuous and real-valued on $[a, b]$ such that $f(a) = f(b)$ and $g(a) = g(b)$. Adding the two equations gives $f(a) + g(a) = f(b) + g(b)$, and thus $(f + g)(a) = (f + g)(b)$, and since calculus informs us that $f + g$ must also be continuous and real-valued on $[a, b]$, we conclude that $f + g \in W$. Therefore W is closed under vector addition.

Finally, for any $f \in W$ and scalar $c \in \mathbb{R}$ we know that cf must be continuous and real-valued, and since $f(a) = f(b) \Rightarrow cf(a) = cf(b) \Rightarrow (cf)(a) = (cf)(b)$, we conclude that $cf \in W$. Therefore W is closed under scalar multiplication. This shows that W is a subspace of $\mathcal{C}[a, b]$.

5 First, $\mathbf{0} = \mathbf{0} + \mathbf{0}$ for $\mathbf{0} \in H, K$, so that $\mathbf{0} \in H + K$ and $H + K \neq \varnothing$.

Let $\mathbf{x}, \mathbf{y} \in H + K$, so $\mathbf{x} = \mathbf{u}_1 + \mathbf{v}_1$ and $\mathbf{y} = \mathbf{u}_2 + \mathbf{v}_2$ for some $\mathbf{u}_1, \mathbf{u}_2 \in H$ and $\mathbf{v}_1, \mathbf{v}_2 \in K$. Now, $\mathbf{u}_1 + \mathbf{u}_2 \in H$ and $\mathbf{v}_1 + \mathbf{v}_2 \in K$ since H and K are vector spaces, and because $\mathbf{x} + \mathbf{y} =$

$(\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{v}_1 + \mathbf{v}_2)$ it follows that $\mathbf{x} + \mathbf{y} \in H + K$ also. Thus $H + K$ is closed under vector addition.

Now let $\mathbf{w} \in H + K$ and $c \in \mathbb{R}$. There exists $\mathbf{u} \in H$ and $\mathbf{v} \in K$ such that $\mathbf{w} = \mathbf{u} + \mathbf{v}$, and since H and K are vector spaces we also have $c\mathbf{u} \in H$ and $c\mathbf{v} \in K$, so that $c\mathbf{w} = c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v} \in H + K$. Therefore $H + K$ is closed under scalar multiplication.

6 Let A be the given matrix. By definition $\operatorname{Nul} A = \{\mathbf{x} : A\mathbf{x} = \mathbf{0}\}$, so the job is to find a basis for the solution set to the homogeneous system with augmented matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 2 & -8 & 1 & 9 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -5 & 1 & 4 & 0 \\ 0 & 1 & -4 & 0 & 6 & 0 \\ 0 & 0 & 0 & 1 & -3 & 0 \end{array}\right]$$

We find that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 5x_3 - 7x_5 \\ 4x_3 - 6x_5 \\ x_3 \\ 3x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix},$$

where x_3 and x_5 are free, and so a basis for $\operatorname{Nul} A$ is the set

$$\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

7 Define the matrix $V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5]$, so the space in question is $\operatorname{Col} V$. With a series of row operations get a new matrix B that is in echelon form:

$$V = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = B.$$

The last two columns of B can each be expressed as linear combinations of the first three columns, so can be discarded. The same relationship must hold for the columns of V, so any basis for $\operatorname{Col} V$ must involve only $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ (i.e. the first three columns of V). But the first three columns of V must be linearly independent since the first three columns of B are. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set that spans $\operatorname{Col} V$, and therefore forms a basis for the space in question.

8 Form the matrix $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}$. Then A is invertible since $\det A = 7 \neq 0$ (Theorem 3.4), and then by the Invertible Matrix Theorem the columns of A (i.e. the vectors in $\mathcal{B}$) form a basis for $\mathbb{R}^2$. As for the change-of-coordinates matrix, by inspection we have $P_{\mathcal{B}} = A$.

9 Let $\mathbf{b}_1(t) = 1 - t^2$, $\mathbf{b}_2(t) = t - t^2$, and $\mathbf{b}_3(t) = 2 - 2t + t^2$. We must find scalars c_1, c_2, c_3 such that $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{p}(t)$. Collecting terms, this gives

$$(c_1 + 2c_3) + (c_2 - 2c_3)t + (-c_1 - c_2 + c_3)t^2 = 3 + t - 6t^2.$$

Matching coefficients, we obtain the system

$$\begin{cases} -c_1 - c_2 + c_3 = -6 \\ \phantom{-c_1 -{}} c_2 - 2c_3 = 1 \\ c_1 \phantom{{}- c_2} + 2c_3 = 3 \end{cases}$$

which has solution $(c_1, c_2, c_3) = (7, -3, -2)$. Therefore

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}.$$

10 Polynomial functions are continuous on $\mathbb{R}$, so each polynomial vector space $\mathbb{P}_n$ is a subspace of the vector space $\mathcal{C}(\mathbb{R})$. Thus, if $\mathcal{C}(\mathbb{R})$ were a finite-dimensional vector space, then by Theorem 4.11 the dimension of $\mathcal{C}(\mathbb{R})$ must be greater than or equal to the dimension of each $\mathbb{P}_n$, which is $n+1$. That is, if $\mathcal{C}(\mathbb{R})$ were finite-dimensional, then $\dim \mathcal{C}(\mathbb{R}) \geq n+1$ for each integer $n \geq 0$, and we would conclude that $\dim \mathcal{C}(\mathbb{R})$ has no upper bound, as with the set of integers. This is a contradiction. Therefore $\mathcal{C}(\mathbb{R})$ must be infinite-dimensional.

11 By the Rank Theorem, $\operatorname{rank} A = 3$, the number of pivot positions in A (which equals the number of pivots in B). Also by the Rank Theorem,

$$\dim \operatorname{Nul} A = (\#\text{ of columns in } A) - \operatorname{rank} A = 6 - 3 = 3.$$

A basis for $\operatorname{Col} A$ is the set of pivot columns of A, so that

$$\text{Basis of } \operatorname{Col} A = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 7 \\ 10 \\ 1 \\ -5 \\ 0 \end{bmatrix} \right\}.$$

A basis for $\operatorname{Row} A$ is the set of nonzero rows of B:

$$\text{Basis of } \operatorname{Row} A = \{(1, 1, -3, 7, 9, -9),\ (0, 1, -1, 3, 4, -3),\ (0, 0, 0, 1, -1, -2)\}.$$

Finally we find a basis for $\operatorname{Nul} A$. The system $A\mathbf{x} = \mathbf{0}$ is equivalent to $B\mathbf{x} = \mathbf{0}$, and if we get B in a reduced row-echelon form C, so that

$$B \sim C = \begin{bmatrix} 1 & 0 & -2 & 0 & 9 & 2 \\ 0 & 1 & -1 & 0 & 7 & 3 \\ 0 & 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

we readily obtain (since $\operatorname{Nul} C = \operatorname{Nul} A$)

$$\operatorname{Nul} A = \operatorname{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -9 \\ -7 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\},$$

with the three vectors in the set being a basis for $\operatorname{Nul} A$.

12 First we get $P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[\, [\mathbf{b}_1]_{\mathcal{C}} \;\; [\mathbf{b}_2]_{\mathcal{C}} \,\big]$. To find $[\mathbf{b}_1]_{\mathcal{C}}$ we find scalars x_1, x_2 such that $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 = \mathbf{b}_1$, which is a system with augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & 8 \end{bmatrix}.$$

Solving gives $x_1 = 3$ and $x_2 = -4$. To find $[\mathbf{b}_2]_{\mathcal{C}}$ we find y_1, y_2 such that $y_1\mathbf{c}_1 + y_2\mathbf{c}_2 = \mathbf{b}_2$, which is a system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & -5 \end{bmatrix}.$$

Solving gives $y_1 = -2$ and $y_2 = 3$. Therefore

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[\, [\mathbf{b}_1]_{\mathcal{C}} \;\; [\mathbf{b}_2]_{\mathcal{C}} \,\big] = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$

13 Setting $\mathcal{B} = \{1 - 3t^2, 2 + t - 5t^2, 1 + 2t\} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$, we find without calculation that

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[\, [\mathbf{b}_1]_{\mathcal{C}} \;\; [\mathbf{b}_2]_{\mathcal{C}} \;\; [\mathbf{b}_3]_{\mathcal{C}} \,\big] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ -3 & -5 & 0 \end{bmatrix}.$$