Math 260 Exam #1 Key (Summer 2021)

1 The system is equivalent to

$$(x_1, x_2) = \left(g - 3x_2, \frac{8-4g}{h-12}\right),$$

from which we conclude that the system has no solution if $g \neq 2$, $h = 12$; a unique solution if $g \in \mathbb{R}$, $h \neq 12$; and an infinite number of solutions if $g = 2$, $h = 12$.

2 The system is equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -9 \\ 0 & 1 & 0 & 4 \end{array}\right]$$

and so the solution set is: $x_1 = -9$, $x_2 = 4$, x_3 free.

3 Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix}$. The system $A\mathbf{x} = \mathbf{b}$ is found by row operations to be inconsistent. That is, there exists no $\mathbf{x} = (x_1, x_2, x_3)$ such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$, and so $\mathbf{b}$ is not a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$.

4 Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Solution is $(x_1, x_2, x_3) = (\frac{3}{5}, -\frac{4}{5}, 1)$.

5 Let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$. With row operations we find that

$$A \sim \begin{bmatrix} -2 & 8 & -5 \\ 0 & -3 & -1 \\ 0 & 0 & 4 \end{bmatrix},$$

which shows that A has a pivot position in every row, and thus by the Invertible Matrix Theorem (IMT) each $\mathbf{b} \in \mathbb{R}^3$ is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. This means $\mathbf{b} \in \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ for each $\mathbf{b} \in \mathbb{R}^3$, so that $\mathbb{R}^3 \subseteq \operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, and therefore $\operatorname{Span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathbb{R}^3$. That is, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

6 We must find integers $x_1, \ldots, x_4$ such that

$$x_1\mathrm{Na_3PO_4} + x_2\mathrm{Ba(NO_3)_2} \longrightarrow x_3\mathrm{Ba_3(PO_4)_2} + x_4\mathrm{NaNO_3}.$$

is a balanced equation. Listing in the order Na, P, O, Ba, N, we need

$$x_1\begin{bmatrix} 3 \\ 1 \\ 4 \\ 0 \\ 0 \end{bmatrix} + x_2\begin{bmatrix} 0 \\ 0 \\ 6 \\ 1 \\ 2 \end{bmatrix} = x_3\begin{bmatrix} 0 \\ 2 \\ 8 \\ 3 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

Collecting all terms on the left side of the equation yields a system with augmented matrix

$$\left[\begin{array}{cccc|c} 3 & 0 & 0 & -1 & 0 \\ 1 & 0 & -2 & 0 & 0 \\ 4 & 6 & -8 & -3 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We find that $x_1 = 2x_3$, $x_2 = 3x_3$, $x_4 = 6x_3$, with x_3 free. Letting $x_3 = 1$ then gives $(x_1, x_2, x_3, x_4) = (2, 3, 1, 6)$.

7 Let A have the vectors as its columns. Now, $A\mathbf{x} = \mathbf{0}$ has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ -1 & 7 & 1 & 0 \\ 3 & 8 & h & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 23 & h-3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -5 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & h-26 & 0 \end{array}\right]$$

The system has nontrivial solutions if and only if $h = 26$, whereupon the IMT implies the columns of A (i.e. the vectors given) are linearly dependent.

8 Since $T(\mathbf{e}_k) = \mathbf{v}_k$ for $k = 1, 2$, the standard matrix is readily obtained:

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} = \begin{bmatrix} -2 & 7 \\ 5 & -3 \end{bmatrix}.$$

9 T linear if and only if $T(\mathbf{0}) = \mathbf{0}$. However, here we have $T(\mathbf{0}) = A\mathbf{0} + \mathbf{b} = \mathbf{b} \neq \mathbf{0}$, and so T is not linear.

10a The standard matrix for T is

$$A = \begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}.$$

Now, $T(\mathbf{x}) = \mathbf{0}$ if and only if $A\mathbf{x} = \mathbf{0}$, but the system $A\mathbf{x} = \mathbf{0}$ has augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 4 & -5 & 0 \\ 3 & -7 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

which has solution set $x_1 = x_3$, $x_2 = x_3$, x_3 free. Thus we find that, for instance, $\mathbf{x} = (1, 1, 1)$ is a solution to $A\mathbf{x} = \mathbf{0}$, so $A\mathbf{x} = \mathbf{0}$ has nontrivial solutions. This immediately implies the column vectors of A are linearly dependent, and therefore T is not one-to-one by Theorem 2 on the back of the exam.

10b From the work in part (a) we see that the 2×2 matrix making up the first two columns of A has a pivot position in every row, and thus the first two columns of A (which constitute a square matrix) must span $\mathbb{R}^2$ by the IMT (which only applies to square matrices). It then follows that the three columns of A itself must span $\mathbb{R}^2$, and therefore T is onto by Theorem 2 on the back of the exam.

11 Let $A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}$ and $B = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$. Suppose $T(\mathbf{x}) = A\mathbf{x}$ and $T(\mathbf{x}) = B\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$. Now, for each $1 \le k \le n$, $T(\mathbf{e}_k) = A\mathbf{e}_k = \mathbf{a}_k$ and $T(\mathbf{e}_k) = B\mathbf{e}_k = \mathbf{b}_k$, so that $\mathbf{a}_k = \mathbf{b}_k$ for each $1 \le k \le n$, and therefore $A = B$. (Note it is not essential to assume that A is the *standard* matrix for T, but the problem is stated as the textbook states it.)

12 Since

$$AB = \begin{bmatrix} 23 & 5k-10 \\ -9 & k+15 \end{bmatrix} \quad \text{and} \quad BA = \begin{bmatrix} 23 & 15 \\ 6-3k & k+15 \end{bmatrix},$$

we have $AB = BA$ only if $5k - 10 = 15$ and $6 - 3k = -9$. Both equations are satisfied if and only if $k = 5$.

13 Using the row-operation algorithm for finding inverses, we obtain

$$C^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

14 Part (k) of the IMT implies E is invertible, so E^{-1} exists and we have

$$EF = I \;\Rightarrow\; E^{-1}(EF) = E^{-1}I \;\Rightarrow\; (E^{-1}E)F = E^{-1} \;\Rightarrow\; IF = E^{-1} \;\Rightarrow\; F = E^{-1}.$$

Now,

$$FE = E^{-1}E = I = EF.$$