Math 260 Exam #3 Key (Summer 2020)

1 The eigenspace consists of all $\mathbf{x} \in \mathbb{R}^3$ such that $\mathbf{A}\mathbf{x} = -2\mathbf{x}$, or equivalently $(\mathbf{A} + 2\mathbf{I})\mathbf{x} = \mathbf{0}$. Solving the system show this to be the subspace

$$\operatorname{Span}\left\{\begin{bmatrix}1 & 1 & 3\end{bmatrix}^\top\right\},$$

and so $\left\{\begin{bmatrix}1 & 1 & 3\end{bmatrix}^\top\right\}$ is a basis.

2 Set $\mathbf{B} = [b_{ij}]$. We're given that, for each $1 \le i \le m$,

$$\sum_{j=1}^{m} b_{ij} = \mu.$$

Let $\mathbf{x}_1 \in \mathbb{R}^m$ have all entries equal to 1. Then

$$\mathbf{B}\mathbf{x}_1 = \begin{bmatrix} \sum_{j=1}^m b_{1j} \\ \vdots \\ \sum_{j=1}^m b_{mj} \end{bmatrix} = \begin{bmatrix} \mu \\ \mu \\ \mu \end{bmatrix} = \mu \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \mu \mathbf{x}_1,$$

which shows that μ is an eigenvalue of $\mathbf{B}$ (with $\mathbf{x}_1$ being an associated eigenvector).

3 This will be $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which works out as

$$\lambda^3 - 2\lambda^2 + 9\lambda + 94 = 0.$$

4 Find bases for the eigenspaces for $\mathbf{A}$. From $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ we get characteristic equation $\lambda^2 - 6\lambda - 16 = 0$, and hence eigenvalues $\lambda = -2, 8$. Solving $(\mathbf{A} + 2\mathbf{I})\mathbf{x} = \mathbf{0}$ gives $\operatorname{Span}\{[3 \ \ 7]^\top\}$, so $\{[3 \ \ 7]^\top\}$ is a basis. Similarly the eigenspace for $\lambda = 8$ has basis $\{[1 \ \ -1]^\top\}$. Therefore we need

$$\mathcal{B} = \left\{ \begin{bmatrix} 3 \\ 7 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}.$$

5 We first get

$$\hat{\mathbf{x}} = \operatorname{proj}_{\mathbf{u}} \mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u} = \frac{10}{73}\begin{bmatrix} 8 \\ -3 \end{bmatrix},$$

and so the distance is

$$\|\mathbf{x} - \hat{\mathbf{x}}\| = \frac{51}{\sqrt{73}} \approx 5.969.$$

6 The linearity of $T(\mathbf{x}) = 2\operatorname{proj}_\ell \mathbf{x} - \mathbf{x}$ will follow from basic properties of the dot product. For scalar c and $\mathbf{x} \in \mathbb{R}^n$,

$$T(c\mathbf{x}) = 2\operatorname{proj}_\ell c\mathbf{x} - c\mathbf{x} = 2\left(\frac{c\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u} - c\mathbf{x} = c\left[2\left(\frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\right)\mathbf{u} - \mathbf{x}\right] = c\left[2\operatorname{proj}_\ell \mathbf{x} - \mathbf{x}\right] = cT(\mathbf{x}),$$

and for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$\begin{aligned}
T(\mathbf{x}+\mathbf{y}) = 2\operatorname{proj}_\ell(\mathbf{x}+\mathbf{y}) - (\mathbf{x}+\mathbf{y}) &= 2\left[\frac{(\mathbf{x}+\mathbf{y})\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right]\mathbf{u} - \mathbf{x} - \mathbf{y} \\
&= 2\left(\frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}} + \frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \mathbf{x} - \mathbf{y} \\
&= \left[2\left(\frac{\mathbf{x}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \mathbf{x}\right] + \left[2\left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} - \mathbf{y}\right] \\
&= [2\operatorname{proj}_\ell \mathbf{x} - \mathbf{x}] + [2\operatorname{proj}_\ell \mathbf{x} - \mathbf{x}] \\
&= T(\mathbf{x}) + T(\mathbf{y}).
\end{aligned}$$

Therefore T is linear.

7 Let

$$\mathbf{v} = \operatorname{proj}_W \mathbf{x} = \sum_{k=1}^{3}\left(\frac{\mathbf{x}\cdot\mathbf{u}_k}{\mathbf{u}_k\cdot\mathbf{u}_k}\right)\mathbf{u}_k = \frac{1}{3}\begin{bmatrix} 4 \\ 11 \\ -7 \\ -4 \end{bmatrix},$$

then set $\mathbf{w} \in W^\top$ to be

$$\mathbf{w} = \mathbf{x} - \mathbf{v} = \frac{1}{3}\begin{bmatrix} -13 \\ 13 \\ 13 \\ 0 \end{bmatrix}.$$

8 Let

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} -8 \\ -4 \\ 6 \\ -2 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ -3 \\ 6 \\ 6 \end{bmatrix}.$$

Set $\mathbf{w}_1 = \mathbf{u}_1$. By the Gram-Schmidt procedure,

$$\mathbf{w}_2 = \mathbf{u}_2 - \left(\frac{\mathbf{u}_2\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\right)\mathbf{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 1 \end{bmatrix},$$

and

$$\mathbf{w}_3 = \mathbf{u}_3 - \left(\frac{\mathbf{u}_3\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\right)\mathbf{w}_1 - \left(\frac{\mathbf{u}_3\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\right)\mathbf{w}_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}.$$

Orthogonal basis for $\operatorname{Col}\mathbf{A}$ is $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$.

9 Best approximation is

$$\hat{p} = \frac{\langle r, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0 + \frac{\langle r, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1 + \frac{\langle r, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2.$$

Now,

$$\langle r, p_0 \rangle = r(-3)p_0(-3) + r(-1)p_0(-1) + r(1)p_0(1) + r(3)p_0(3) = 0,$$

and similarly $\langle r, p_1 \rangle = 164$, $\langle r, p_2 \rangle = 0$, $\langle p_1, p_1 \rangle = 20$. Note we don't need $\langle p_2, p_2 \rangle$. We finally obtain

$$\hat{p}(t) = \frac{164}{20} p_1(t) = \frac{41}{5} t.$$

10 For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and scalar c, we find using linearity and dot product properties that

$$\langle \mathbf{u}, \mathbf{v} \rangle = T(\mathbf{u}) \cdot T(\mathbf{v}) = T(\mathbf{v}) \cdot T(\mathbf{u}) = \langle \mathbf{v}, \mathbf{u} \rangle,$$

and

$$\begin{aligned} \langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle &= T(\mathbf{u} + \mathbf{v}) \cdot T(\mathbf{w}) = [T(\mathbf{u}) + T(\mathbf{v})] \cdot T(\mathbf{w}) \\ &= T(\mathbf{u}) \cdot T(\mathbf{w}) + T(\mathbf{v}) \cdot T(\mathbf{w}) = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle, \end{aligned}$$

and

$$\langle c\mathbf{u}, \mathbf{v} \rangle = T(c\mathbf{u}) \cdot T(\mathbf{v}) = cT(\mathbf{u}) \cdot T(\mathbf{v}) = c[T(\mathbf{u}) \cdot T(\mathbf{v})] = c\langle \mathbf{u}, \mathbf{v} \rangle,$$

and

$$\langle \mathbf{u}, \mathbf{u} \rangle = T(\mathbf{u}) \cdot T(\mathbf{u}) = \|T(\mathbf{u})\|^2 \geq 0.$$

Next, suppose that $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. Then $T(\mathbf{u}) \cdot T(\mathbf{u}) = \|T(\mathbf{u})\|^2 = 0$, which implies $T(\mathbf{u}) = \mathbf{0}$, and since T is one-to-one we conclude that $\mathbf{u} = \mathbf{0}$. Finally, if $\mathbf{u} = \mathbf{0}$, then $T(\mathbf{u}) = T(\mathbf{0}) = \mathbf{0}$, and so $\langle \mathbf{u}, \mathbf{u} \rangle = \mathbf{0} \cdot \mathbf{0} = 0$.

11 Let $\mathbf{u}_1 = 3$, $\mathbf{u}_2 = 2t$, $\mathbf{u}_3 = t^2$. Set $\mathbf{w}_1 = \mathbf{u}_1 = 3$. By the Gram-Schmidt procedure,

$$\mathbf{w}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 = 2t - \frac{\int_{-3}^{3} 6t \, dt}{\int_{-3}^{3} 9 \, dt} (3) = 2t,$$

and

$$\mathbf{w}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 - \frac{\langle \mathbf{u}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w}_2 = t^2 - \frac{\int_{-3}^{3} 3t^2 \, dt}{\int_{-3}^{3} 9 \, dt} (3) - \frac{\int_{-3}^{3} 2t^3 \, dt}{\int_{-3}^{3} 4t^2 \, dt} (2t) = t^2 - 3.$$

$\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is an orthogonal basis.