Math 260 Exam #2 Key (Summer 2020)

1 We have:

$$\begin{vmatrix} a & b & c \\ d+4g & e+4h & f+4i \\ g & h & i \end{vmatrix} = -9 \quad \Rightarrow \quad \begin{vmatrix} g & h & i \\ d+4g & e+4h & f+4i \\ a & b & c \end{vmatrix} = 9,$$

and hence

$$\begin{vmatrix} -3g & -3h & -3i \\ d+4g & e+4h & f+4i \\ -2a & -2b & -2c \end{vmatrix} = (-3)(-2)9 = 54.$$

2 Using the property $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$,

$$\det(\mathbf{A}^7) = 0 \quad \Rightarrow \quad (\det \mathbf{A})^7 = 0 \quad \Rightarrow \quad \det \mathbf{A} = 0,$$

and so $\mathbf{A}$ is not invertible by the Invertible Matrix Theorem.

3 The system is $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 1 & 3 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}.$$

Expanding along the 3rd column we have

$$\det \mathbf{A} = \begin{vmatrix} -1 & 0 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = -1 - 2(1-9) = 15,$$

so $\det \mathbf{A} \neq 0$ and by Cramer's Rule

$$x_1 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 4 & 3 & 1 \\ 2 & 0 & 2 \\ 2 & 1 & 0 \end{vmatrix} = \frac{1}{15}(6) = \frac{2}{5}$$

$$x_2 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 1 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & 2 & 0 \end{vmatrix} = \frac{1}{15}(12) = \frac{4}{5}$$

$$x_3 = \frac{1}{\det \mathbf{A}} \begin{vmatrix} 1 & 3 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & 2 \end{vmatrix} = \frac{1}{15}(18) = \frac{6}{5}$$

Therefore the solution to the system is $\left(\frac{2}{5}, \frac{4}{5}, \frac{6}{5}\right)$.

4 Take the absolute value of the determinant

$$\begin{vmatrix} 1 & -2 & -1 \\ 3 & 0 & 3 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & -1 \\ 3 & 0 & 3 \\ 1 & 0 & -2 \end{vmatrix} = -18,$$

to get a volume of 18.

5 We find that

$$W = s\begin{bmatrix}1\\1\\2\\0\end{bmatrix} + t\begin{bmatrix}3\\-1\\-7\\4\end{bmatrix} = \text{Span}\left\{\begin{bmatrix}1\\1\\2\\0\end{bmatrix}, \begin{bmatrix}3\\-1\\-7\\4\end{bmatrix}\right\},$$

and so W is a subspace of $\mathbb{R}^4$ by virtue of the fact that it equals the span of two vectors in $\mathbb{R}^4$. (See Theorem 1 on p. 196.)

6 Let W denote the set. The zero function on $[a,b]$, which we'll denote here by $\mathbf{0}$, is certainly a continuous real-valued function, and since $\mathbf{0}(a) = 0 = \mathbf{0}(b)$ it follows that $\mathbf{0} \in W$.

Next, suppose $f, g \in W$, so f and g are continuous and real-valued on $[a,b]$ such that $f(a) = f(b)$ and $g(a) = g(b)$. Adding the two equations gives $f(a) + g(a) = f(b) + g(b)$, and thus $(f+g)(a) = (f+g)(b)$, and since calculus informs us that $f+g$ must also be continuous and real-valued on $[a,b]$, we conclude that $f+g \in W$. Therefore W is closed under vector addition.

Finally, for any $f \in W$ and scalar $c \in \mathbb{R}$ we know that cf must be continuous and real-valued, and since $f(a) = f(b) \Rightarrow cf(a) = cf(b) \Rightarrow (cf)(a) = (cf)(b)$, we conclude that $cf \in W$. Therefore W is closed under scalar multiplication. This shows that W is a subspace of $\mathcal{C}[a,b]$.

7 W lacks the zero vector: letting $r = s = t = 0$ gives the falsehood $2 = 0$ in the equation that defines W. This immediately implies W is not a vector space.

8 Let $\mathbf{A}$ be the given matrix. By definition $\text{Nul}\,\mathbf{A} = \{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{0}\}$, so the job is to find a basis for the solution set to the homogeneous system with augmented matrix

$$\left[\begin{array}{ccccc|c}1 & 0 & -5 & 1 & 4 & 0\\0 & 1 & -4 & 0 & 6 & 0\\0 & 2 & -8 & 1 & 9 & 0\end{array}\right] \sim \left[\begin{array}{ccccc|c}1 & 0 & -5 & 1 & 4 & 0\\0 & 1 & -4 & 0 & 6 & 0\\0 & 0 & 0 & 1 & -3 & 0\end{array}\right]$$

We find that

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix} = \begin{bmatrix}5x_3 - 7x_5\\4x_3 - 6x_5\\x_3\\3x_5\\x_5\end{bmatrix} = x_3\begin{bmatrix}5\\4\\1\\0\\0\end{bmatrix} + x_5\begin{bmatrix}-7\\-6\\0\\3\\1\end{bmatrix},$$

where x_3 and x_5 are free, and so a basis for $\text{Nul}\,\mathbf{A}$ is the set

$$\left\{\begin{bmatrix}5\\4\\1\\0\\0\end{bmatrix}, \begin{bmatrix}-7\\-6\\0\\3\\1\end{bmatrix}\right\}.$$

9 Define the matrix $\mathbf{V} = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4\ \mathbf{v}_5]$, so the space in question is $\operatorname{Col}\mathbf{V}$. With a series of row operations get a new matrix $\mathbf{B}$ that is in echelon form:

$$\mathbf{V} = \begin{bmatrix} 1 & -2 & 6 & 5 & 0 \\ 0 & 1 & -1 & -3 & 3 \\ 0 & -1 & 2 & 3 & -1 \\ 1 & 1 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 & 8 \\ 0 & 1 & 0 & -3 & 5 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}.$$

The last two columns of $\mathbf{B}$ can each be expressed as linear combinations of the first three columns, so can be discarded. The same relationship must hold for the columns of $\mathbf{V}$, so any basis for $\operatorname{Col}\mathbf{V}$ must involve only $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ (i.e. the first three columns of $\mathbf{V}$). But the first three columns of $\mathbf{V}$ must be linearly independent since the first three columns of $\mathbf{B}$ are. Thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set that spans $\operatorname{Col}\mathbf{V}$, and therefore forms a basis for the space in question.

10 This one is easy, since the columns of the matrix are given:

$$P_{\mathcal{B}} = \begin{bmatrix} 3 & 2 & 8 \\ -1 & 0 & -2 \\ 4 & -5 & 7 \end{bmatrix}.$$

11 Let $\mathbf{b}_1(t) = 1 - t^2$, $\mathbf{b}_2(t) = t - t^2$, and $\mathbf{b}_3(t) = 2 - 2t + t^2$. We must find scalars c_1, c_2, c_3 such that $c_1\mathbf{b}_1 + c_2\mathbf{b}_2 + c_3\mathbf{b}_3 = \mathbf{p}(t)$. Collecting terms, this gives

$$(c_1 + 2c_3) + (c_2 - 2c_3)t + (-c_1 - c_2 + c_3)t^2 = 3 + t - 6t^2.$$

Matching coefficients, we obtain the system

$$\begin{cases} -c_1 - c_2 + c_3 = -6 \\ \qquad\ \ c_2 - 2c_3 = 1 \\ \ c_1 \qquad\ + 2c_3 = 3 \end{cases}$$

which has solution $(c_1, c_2, c_3) = (7, -3, -2)$. Therefore

$$[\mathbf{p}(t)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix}.$$

12 See the discussion on page 230 of the text: since $\mathbf{A}$ is already in echelon form, the dimension of the null space of $\mathbf{A}$ equals the number of nonpivot columns, while the dimension of the column space of $\mathbf{A}$ equals the number of pivot columns. Hence $\operatorname{Nul}\mathbf{A} = 3$ and $\operatorname{Col}\mathbf{A} = 3$.

Alternatively: for $m \times n$ matrix $\mathbf{A}$ use the Rank Theorem on page 235, which states that

$$\dim\operatorname{Col}\mathbf{A} + \dim\operatorname{Nul}\mathbf{A} = n.$$

13 Polynomial functions are continuous on $\mathbb{R}$, so each polynomial vector space $\mathbb{P}_n$ is a subspace of the vector space $\mathcal{C}(\mathbb{R})$. Thus, if $\mathcal{C}(\mathbb{R})$ were a finite-dimensional vector space, then by Theorem

11 on page 229 of the text the dimension of $\mathcal{C}(\mathbb{R})$ must be greater than or equal to the dimension of each $\mathbb{P}_n$, which is $n+1$. That is, if $\mathcal{C}(\mathbb{R})$ were finite-dimensional, then $\dim \mathcal{C}(\mathbb{R}) \geq n+1$ for each integer $n \geq 0$, and we would conclude that $\dim \mathcal{C}(\mathbb{R})$ has no upper bound, as with the set of integers. This is a contradiction. Therefore $\mathcal{C}(\mathbb{R})$ must be infinite-dimensional.

14 Let $\mathbf{A}$ be $m \times n$ with $m > n$ (i.e. a matrix with more rows than columns). Clearly $\dim \operatorname{Col} \mathbf{A} \leq n$ and $\dim \operatorname{Row} \mathbf{A} \leq m$ must hold, but by the Rank Theorem (p. 235),

$$\operatorname{rank} \mathbf{A} = \dim \operatorname{Col} \mathbf{A} = \dim \operatorname{Row} \mathbf{A},$$

so since $n < m$ we conclude that $\mathbf{A}$ can have full rank if and only if $\dim \operatorname{Col} \mathbf{A} = n$. To finish the argument we now just need to prove that $\dim \operatorname{Col} \mathbf{A} = n$ if and only if the columns of $\mathbf{A}$ are linearly independent.

Suppose $\dim \operatorname{Col} \mathbf{A} = n$. Because $\operatorname{Col} \mathbf{A}$ is the span of the n columns of $\mathbf{A}$, we conclude by the Basis Theorem (p. 229) that the columns of $\mathbf{A}$ are linearly independent.

For the converse, suppose the columns of $\mathbf{A}$ are linearly independent. Then the n columns of $\mathbf{A}$ form a basis for $\operatorname{Col} \mathbf{A}$ by the Basis Theorem, implying $\dim \operatorname{Col} \mathbf{A} = n$.

15 First we get $P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[\, [\mathbf{b}_1]_\mathcal{C} \;\; [\mathbf{b}_2]_\mathcal{C} \,\big]$. To find $[\mathbf{b}_1]_\mathcal{C}$ we find scalars x_1, x_2 such that $x_1\mathbf{c}_1 + x_2\mathbf{c}_2 = \mathbf{b}_1$, which is a system with augmented matrix

$$\begin{bmatrix} 1 & 1 & -1 \\ 4 & 1 & 8 \end{bmatrix}.$$

Solving gives $x_1 = 3$ and $x_2 = -4$. To find $[\mathbf{b}_2]_\mathcal{C}$ we find y_1, y_2 such that $y_1\mathbf{c}_1 + y_2\mathbf{c}_2 = \mathbf{b}_2$, which is a system with augmented matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 4 & 1 & -5 \end{bmatrix}.$$

Solving gives $y_1 = -2$ and $y_2 = 3$. Therefore

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \big[\, [\mathbf{b}_1]_\mathcal{C} \;\; [\mathbf{b}_2]_\mathcal{C} \,\big] = \begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix} \quad \text{and} \quad P_{\mathcal{B}\leftarrow\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}.$$