MATH 260 EXAM #1 KEY (SUMMER 2020)

1 3 times row 1 added to row 2 gives

$$\begin{bmatrix} 2 & -3 & h \\ 0 & 0 & 5+3h \end{bmatrix},$$

and so the system is consistent if and only if $5 + 3h = 0$, or $h = -\frac{5}{3}$.

2 Solution set may be written in vector notation as

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \middle| \, x = \tfrac{2}{3}y - \tfrac{4}{3}, \ z = -2y - 3, \ y \text{ is free} \right\} = \left\{ \begin{bmatrix} 2/3 \\ 1 \\ -2 \end{bmatrix} y + \begin{bmatrix} -4/3 \\ 0 \\ -3 \end{bmatrix} \middle| \, y \text{ is free} \right\},$$

though other forms are possible.

3 The problem is to find $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 = \mathbf{b}$, where $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are the columns of $\mathbf{A}$. This is a system of linear equations having augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 1 & -2 & 5 & 9 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -6 & 11 \\ 0 & 3 & 7 & -5 \\ 0 & 0 & 11 & -2 \end{array}\right].$$

We find a solution: $x_1 = \frac{245}{33}$, $x_2 = -\frac{41}{33}$, $x_3 = -\frac{2}{11}$. Therefore $\mathbf{b}$ is indeed a linear combination of the columns of $\mathbf{A}$.

4 This is a matter of finding all values of h such that there exist $x_1, x_2 \in \mathbb{R}$ for which $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 = \mathbf{y}$. This is a system with augmented matrix

$$\begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ -2 & 8 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 2 & 2h-3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & h \\ 0 & 1 & -5 \\ 0 & 0 & 2h+7 \end{bmatrix}.$$

The system is only consistent if $2h + 7 = 0$, or $h = -\frac{7}{2}$.

5 Augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -3 & -1 & 2 & 1 \\ 0 & 5 & 3 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 5 & 5 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Solution is $(x, y, z) = (\frac{3}{5}, -\frac{4}{5}, 1)$.

6 Throughout, let $\mathbf{c}$ denote a solution to $\mathbf{Ax} = \mathbf{b}$, which exists by hypothesis.

Suppose $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution $\boldsymbol{\xi}$, so $\mathbf{A}\boldsymbol{\xi} = \mathbf{0}$ for $\boldsymbol{\xi} \neq \mathbf{0}$. Now, $\boldsymbol{\xi} + \mathbf{c} \neq \mathbf{c}$, and yet

$$\mathbf{A}(\boldsymbol{\xi} + \mathbf{c}) = \mathbf{A}\boldsymbol{\xi} + \mathbf{Ac} = \mathbf{0} + \mathbf{b} = \mathbf{b},$$

which shows that $\mathbf{Ax} = \mathbf{b}$ has solutions $\mathbf{c}$ and $\boldsymbol{\xi}$; that is, any solution to $\mathbf{Ax} = \mathbf{b}$ is not unique.

Conversely, suppose the solution to $\mathbf{Ax}=\mathbf{b}$ is not unique, which is to say in addition to $\mathbf{c}$ there is another solution $\boldsymbol{\xi}\neq\mathbf{c}$. It follows that $\boldsymbol{\xi}-\mathbf{c}\neq\mathbf{0}$, and yet

$$\mathbf{A}(\boldsymbol{\xi}-\mathbf{c})=\mathbf{A}\boldsymbol{\xi}-\mathbf{Ac}=\mathbf{b}-\mathbf{b}=\mathbf{0},$$

which shows that $\mathbf{Ax}=\mathbf{0}$ does not only have the trivial solution.

7 We must find integers $x_1,\ldots,x_4$ such that

$$x_1\mathrm{Na_3PO_4}+x_2\mathrm{Ba(NO_3)_2}\longrightarrow x_3\mathrm{Ba_3(PO_4)_2}+x_4\mathrm{NaNO_3}.$$

is a balanced equation. Listing in the order Na, P, O, Ba, N, we need

$$x_1\begin{bmatrix}3\\1\\4\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\0\\6\\1\\2\end{bmatrix}=x_3\begin{bmatrix}0\\2\\8\\3\\0\end{bmatrix}+x_4\begin{bmatrix}1\\0\\3\\0\\1\end{bmatrix}.$$

Collecting all terms on the left side of the equation yields a system with augmented matrix

$$\left[\begin{array}{cccc|c}3&0&0&-1&0\\1&0&-2&0&0\\4&6&-8&-3&0\\0&1&-3&0&0\\0&2&0&1&0\end{array}\right]\sim\left[\begin{array}{cccc|c}1&0&-2&0&0\\0&1&-3&0&0\\0&0&6&-1&0\\0&0&0&0&0\\0&0&0&0&0\end{array}\right].$$

We find that $x_1=2x_3$, $x_2=3x_3$, $x_4=6x_3$, with x_3 free. Letting $x_3=1$ then gives $(x_1,x_2,x_3,x_4)=(2,3,1,6)$.

8 This is disproven with the counterexample

$$\mathbf{v}_1=\begin{bmatrix}1\\0\\0\\0\end{bmatrix},\quad\mathbf{v}_2=\begin{bmatrix}0\\1\\0\\0\end{bmatrix},\quad\mathbf{v}_3=\begin{bmatrix}0\\0\\1\\0\end{bmatrix},\quad\mathbf{v}_4=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}.$$

9 We have

$$T((-1,-1)+(1,1))=T(0,0)=(0,0)$$

and

$$T(-1,-1)+T(1,1)=(-2,3)+(2,3)=(0,6),$$

so $T((-1,-1)+(1,1))\neq T(-1,-1)+T(1,1)$.

10a Find $x_1,x_2\in\mathbb{R}$ such that

$$\begin{cases}x_1-2x_2=-1\\-x_1+3x_2=4\\3x_1-2x_2=9\end{cases}$$

Solving gives $(x_1,x_2)=(5,3)$.

10b Suppose $T(\mathbf{x}) = \mathbf{0}$, giving the system

$$\begin{cases} x_1 - 2x_2 = 0 \\ -x_1 + 3x_2 = 0 \\ 3x_1 - 2x_2 = 0 \end{cases}$$

The only solution is $(x_1, x_2) = (0, 0)$. Thus $T(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} = \mathbf{0}$. Theorem 11 on page 77 of the text then implies that T is one-to-one.

10c The standard matrix $\mathbf{A}$ corresponding to T must be 3×2, so $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2]$ for some $\mathbf{a}_1, \mathbf{a}_2 \in \mathbb{R}^3$. This implies $\mathbf{A}$ cannot have a pivot position in every row. By Theorem 4 on page 37 it follows that the columns of $\mathbf{A}$ cannot span $\mathbb{R}^3$, and so T is not onto by Theorem 12a on page 78.

11 Find a, b, c, d such that

$$\begin{bmatrix} 3 & -4 \\ -1 & 8 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This gives

$$\begin{bmatrix} 3a - 4c & 3b - 4d \\ -a + 8c & -b + 8d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

yielding the system

$$\begin{cases} 3a - 4c = 0 \\ 3b - 4d = 0 \\ -a + 8c = 0 \\ -b + 8d = 0 \end{cases}$$

The only solution is $a = b = c = d = 0$, so no matrix $\mathbf{B}$ exists with the required properties.

12 Using known properties of matrix arithmetic, since $\mathbf{D}^{-1}$ exists:

$$(\mathbf{B} - \mathbf{C})\mathbf{D} = \mathbf{O} \;\Rightarrow\; (\mathbf{B} - \mathbf{C})\mathbf{D}\mathbf{D}^{-1} = \mathbf{O}\mathbf{D}^{-1} \;\Rightarrow\; (\mathbf{B} - \mathbf{C})\mathbf{I} = \mathbf{O}$$
$$\Rightarrow\; \mathbf{B} - \mathbf{C} = \mathbf{O} \;\Rightarrow\; \mathbf{B} = \mathbf{C}.$$

13 We have

$$T(x_1, x_2) = \begin{bmatrix} 6x_1 - 8x_2 \\ -5x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

and so the standard matrix for T is

$$\mathbf{A} = \begin{bmatrix} 6 & -8 \\ -5 & 7 \end{bmatrix}.$$

A theorem informs us that T is invertible if and only if $\mathbf{A}$ is invertible. The inverse of $\mathbf{A}$ indeed exists: omitting the computations, we have

$$\mathbf{A}^{-1} = \begin{bmatrix} 7/2 & 4 \\ 5/2 & 3 \end{bmatrix}.$$

With this we find a formula for T^{-1}:

$$T^{-1}(x_1, x_2) = \left(\tfrac{7}{2}x_1 + 4x_2, \tfrac{5}{2}x_1 + 3x_2\right).$$

14 Form a 3×3 matrix $\mathbf{A}$ with the given vectors as its columns. By the Invertible Matrix Theorem on page 114, if $\mathbf{A}$ is invertible then the columns of $\mathbf{A}$ (and hence the vectors themselves) form a linearly independent set that spans $\mathbb{R}^3$; that is, the vectors form a basis for $\mathbb{R}^3$. Is $\mathbf{A}$ invertible? We can verify that it is by computing it directly. Starting with

$$\left[\begin{array}{rrr|rrr} 1 & -5 & 7 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ -2 & 2 & -5 & 0 & 0 & 1 \end{array}\right],$$

transform the matrix $\mathbf{A}$ at left via row operations until the 3×3 identity matrix is obtained:

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -1/4 & 11/20 & -7/20 \\ 0 & 1 & 0 & -1/4 & -9/20 & -7/20 \\ 0 & 0 & 1 & 0 & -2/5 & -1/5 \end{array}\right].$$

At right is $\mathbf{A}^{-1}$, so $\mathbf{A}$ is invertible.

15 The task is to find $x_1, x_2 \in \mathbb{R}$ such that $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = \mathbf{x}$. This is a linear system with augmented matrix

$$\begin{bmatrix} -3 & 7 & 11 \\ 1 & 5 & 0 \\ -4 & -6 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix},$$

and we find that $x_1 = -5/2$ and $x_2 = 1/2$. Therefore

$$[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 1/2 \end{bmatrix}.$$

16 By the Rank Theorem,

$$\operatorname{rank}\mathbf{A} = n - \dim \operatorname{Nul}\mathbf{A} = 6 - 2 = 4.$$