Math 260 Exam #2 Key (Summer 2018)

1 Let S_1 and S_2 be convex sets in V. Let $\mathbf{u}, \mathbf{v} \in S_1 \cap S_2$. Since S_1 is convex and $\mathbf{u}, \mathbf{v} \in S_1$, we find that $[\mathbf{u}, \mathbf{v}] \subseteq S_1$. Similarly, since S_2 is convex and $\mathbf{u}, \mathbf{v} \in S_2$, we have $[\mathbf{u}, \mathbf{v}] \subseteq S_2$. Therefore $[\mathbf{u}, \mathbf{v}] \subseteq S_1 \cap S_2$ and we conclude that $S_1 \cap S_2$ is convex.

2 Let $\mathbf{u}_1 = [-6, 1]$ and $\mathbf{u}_2 = [-3, 2]$. Find c_1 and c_2 such that $c_1\mathbf{u}_1 + c_2\mathbf{u}_2 = \mathbf{x}$, This results in the system

$$\begin{cases} -6c_1 - 3c_2 = 4 \\ \quad c_1 + 2c_2 = -8 \end{cases}$$

Solving yields $c_1 = 16/9$ and $c_2 = -44/9$, so the $\mathcal{B}$-coordinates of $\mathbf{x}$ are $[\mathbf{x}]_{\mathcal{B}} = \left[\frac{16}{9}, -\frac{44}{9}\right]$.

3 Let $f(x) = e^t$ and $g(t) = \ln t$. Suppose $c_1 f + c_2 g = 0$ on $(0, \infty)$. Then in particular we have $c_1 f(1) + c_2 g(1) = 0$ and $c_1 f(2) + c_2 g(2) = 0$. The first equation gives $c_1 e + c_2 \ln(1) = 0$, and hence $c_1 = 0$. The second equation then becomes $c_2 g(2) = 0$, or $c_2 \ln 2 = 0$, and hence $c_2 = 0$. Therefore f and g are linearly independent on $(0, \infty)$.

4 With the elementary row operations $-9r_1 + r_2$, $r_1 + r_3$, and $-4r_1 + r_4$ we obtain

$$\mathbf{M} \sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 16 & 1 & -36 \\ 0 & -2 & -1 & 12 \\ 0 & 0 & 2 & -28 \end{bmatrix}.$$

Next we perform the operations $r_2 \leftrightarrow r_3$, $8r_2 + r_3$, and $\frac{2}{7}r_3 + r_4$ to get an echelon form:

$$\mathbf{M} \sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & -2 & -1 & 12 \\ 0 & 0 & -7 & 60 \\ 0 & 0 & 0 & -76/7 \end{bmatrix}.$$

The rank of $\mathbf{M}$ is equal to the number of pivots in a row-equivalent row-echelon form, and so $\operatorname{rank}(\mathbf{M}) = 4$.

5 Find a and b such that

$$a(1, 1) + b(-1, 3) = (0, 1),$$

so $a - b = 0$ and $a + 3b = 1$. We find $a = b = 1/4$. Now, by linearity,

$$L(0, 1) = L\big(\tfrac{1}{4}(1, 1) + \tfrac{1}{4}(-1, 3)\big) = \tfrac{1}{4}L(1, 1) + \tfrac{1}{4}L(-1, 3) = \tfrac{1}{4}(2, -1) + \tfrac{1}{4}(1, 2) = \big(\tfrac{3}{4}, \tfrac{1}{4}\big).$$

6 Suppose $\sum_{k=1}^{n} c_k\mathbf{v}_k = \mathbf{0}$. Then

$$\mathbf{0} = L(\mathbf{0}) = L\left(\sum_{k=1}^{n} c_k\mathbf{v}_k\right) = \sum_{k=1}^{n} c_k L(\mathbf{v}_k) = \sum_{k=1}^{n} c_k\mathbf{w}_k$$

by linearity properties, and since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent we conclude that $c_k = 0$ for all k.

7 By the Rank-Nullity Theorem, $\dim(\operatorname{Img} L)+\dim(\operatorname{Ker} L) = \dim V$. Since $\operatorname{Img} L$ is a subspace of W, $\dim(\operatorname{Img} L) \leq \dim W$. Thus $\dim(\operatorname{Img} L) < \dim V$, and it follows that $\dim(\operatorname{Ker} L) > 0$. That is, $\dim(\operatorname{Ker} L) \geq 1$, and we conclude that $\operatorname{Ker} L \neq \{\mathbf{0}\}$.

8 Let

$$\mathbf{v}_1^\top = [1, 1, -2, 3, 4], \quad \mathbf{v}_2^\top = [1, 0, 0, 2, 0,], \quad \mathbf{v}_3^\top = [0, 1, 0, 1, 0],$$

and let $\mathbf{A}$ be the 3×5 matrix with row vectors $\mathbf{v}_1^\top$, $\mathbf{v}_2^\top$, $\mathbf{v}_3^\top$. Three column vectors of $\mathbf{A}$ are

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix},$$

which are linearly independent in $\mathbb{R}^3$, and so $\operatorname{rank} \mathbf{A} = 3$. Let U be the subspace in question, so

$$U = \{\mathbf{x} \in \mathbb{R}^5 : \mathbf{v}_1 \cdot \mathbf{x} = 0,\ \mathbf{v}_2 \cdot \mathbf{x} = 0,\ \mathbf{v}_3 \cdot \mathbf{x} = 0\} = \{\mathbf{x} \in \mathbb{R}^5 : \mathbf{A}\mathbf{x} = \mathbf{0}\}.$$

By the Matrix Rank-Nullity Theorem

$$\dim U = \dim(\operatorname{Nul} \mathbf{A}) = \operatorname{nullity} \mathbf{A} = \dim \mathbb{R}^5 - \operatorname{rank} \mathbf{A} = 5 - 3 = 2.$$

9 If $\mathcal{E}_4$ and $\mathcal{E}_3$ are the standard bases for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively, then

$$[L]_{\mathcal{E}_4\mathcal{E}_3} = \Big[\big[L(\mathbf{e}_1)\big]_{\mathcal{E}_3} \quad \cdots \quad \big[L(\mathbf{e}_4)\big]_{\mathcal{E}_3} \Big] = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{bmatrix}.$$

10 Suppose $L(x, y) = [0, 0, 0]$. Then $8x + y = 0$, $3x - 5y = 0$, and $x - 4y = 0$ which are only satisfied if $x = y = 0$. Thus $\operatorname{Ker} L = \{[0, 0]\}$, and so the $\dim(\operatorname{Ker} L) = 0$. The Rank-Nullity Theorem states that

$$\dim(\operatorname{Ker} L) + \dim(\operatorname{Img} L) = \dim(\mathbb{R}^{1\times 2}),$$

so

$$\dim(\operatorname{Img} L) = \dim(\mathbb{R}^{1\times 2}) - \dim(\operatorname{Ker} L) = 2 - 0 = 2.$$

Now, $\operatorname{Img} L$ is a subspace of $\mathbb{R}^{1\times 3}$, and since $\dim(\operatorname{Img} L) = 2$ while $\dim(\mathbb{R}^{1\times 3}) = 3$, it follows that $\operatorname{Img} L \neq \mathbb{R}^{1\times 3}$. Hence the mapping L is not surjective, and therefore is not invertible.

11 Setting $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ and $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2)$, the transition matrix is $\mathbf{I}_{\mathcal{BC}} = \big[[\mathbf{v}_1]_\mathcal{C} \quad [\mathbf{v}_2]_\mathcal{C}\big]$. Here $[\mathbf{v}_1]_\mathcal{C} = [a\ b]^\top$ for a, b such that $a\mathbf{w}_1 + b\mathbf{w}_2 = \mathbf{v}_1$, and $[\mathbf{v}_2]_\mathcal{C} = [c\ d]^\top$ for c, d such that $c\mathbf{w}_1 + d\mathbf{w}_2 = \mathbf{v}_2$. Solving the two systems of equations

$$\begin{cases} a + b = 1 \\ -a + 3b = 1 \end{cases} \qquad \begin{cases} c + d = 2 \\ -c + 3d = 0 \end{cases}$$

yields $a = \frac{1}{2}$, $b = \frac{1}{2}$, $c = \frac{3}{2}$, and $d = \frac{1}{2}$, and so

$$\mathbf{I}_{\mathcal{BC}} = \begin{bmatrix} 1/2 & 3/2 \\ 1/2 & 1/2 \end{bmatrix}.$$