Math 260 Exam #1 Key (Summer 2018)

1a Both $\pm 2\mathbf{u}/\|\mathbf{u}\|$ will work, where $2\mathbf{u}/\|\mathbf{u}\| = \frac{2}{\sqrt{90}}[3,-1,4,8]$.

1b Find any x and y such that $[x,6,y,-3]\cdot\mathbf{u}=0$, where

$$[x,6,y,-3]\cdot\mathbf{u}=0 \quad\Rightarrow\quad 3x+4y=30.$$

A convenient choice would be $x=10$ and $y=0$, so that $[10,6,0,-3]$ is orthogonal to $\mathbf{u}$.

2a By definition,

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v} = -\tfrac{17}{6}[-2,-1,1].$$

2b Here we go:

$$\operatorname{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u} = -\tfrac{17}{54}[5,5,-2].$$

2c Let θ be the angle. By definition,

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = -\frac{17}{\sqrt{54}\sqrt{6}} = -\frac{17}{\sqrt{324}}.$$

2d Let θ be the angle between $\operatorname{proj}_{\mathbf{v}}\mathbf{u}$ and $\operatorname{proj}_{\mathbf{u}}\mathbf{v}$, and let φ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Using properties of the dot product, we find that

$$\cos\theta = \frac{\operatorname{proj}_{\mathbf{v}}\mathbf{u}\cdot\operatorname{proj}_{\mathbf{u}}\mathbf{v}}{\|\operatorname{proj}_{\mathbf{v}}\mathbf{u}\|\|\operatorname{proj}_{\mathbf{u}}\mathbf{v}\|} = \frac{\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}\cdot\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}}{\left\|\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}\right\|\left\|\left(\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}\right\|} = \frac{\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)^2(\mathbf{v}\cdot\mathbf{u})}{\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)^2\|\mathbf{v}\|\|\mathbf{u}\|} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \cos\varphi.$$

Now, since it is known that θ and φ must both be angles in the interval $[0,\pi]$ where cosine is one-to-one, we conclude that $\theta=\varphi$.

3 A parametrization of the line segment $[\mathbf{p},\mathbf{q}]$ is

$$\mathbf{x}(t) = (1-t)\mathbf{p}+t\mathbf{q}, \quad t\in[0,1].$$

The point $1/3$ of the way from $\mathbf{p}$ to $\mathbf{q}$ on $[\mathbf{p},\mathbf{q}]$ is

$$\mathbf{x}\left(\tfrac{1}{3}\right) = \tfrac{2}{3}\mathbf{p}+\tfrac{1}{3}\mathbf{q} = \left[\tfrac{10}{3},\tfrac{5}{3},-\tfrac{7}{3}\right].$$

4 $\mathbf{x}(t) = \mathbf{p}+t(\mathbf{q}-\mathbf{p}) = [1,0,-1]+t[1,2,-2] = [1+t,2t,-1-2t]$ for $t\in\mathbb{R}$.

5 Points (x,y,z) that lie on both planes must satisfy the system

$$\begin{cases} x-2y+z=0 \\ 2x-3y+z=6 \end{cases}$$

The first equation gives $z = 2y - x$, which when put into the second equation gives $y = x - 6$. Putting this back into $z = 2y - x$ gives $z = x - 12$. The solution set of the system is

$$\{[x, x-6, x-12] : x \in \mathbb{R}\}.$$

Thus a parametric equation for the line of intersection of the two planes is

$$\mathbf{x}(t) = [t, t-6, t-12], \quad t \in \mathbb{R}.$$

6 Since matrix multiplication is associative,

$$\mathbf{BAC} = \mathbf{B}(\mathbf{AC}) = \mathbf{B}\begin{bmatrix} a+20 \\ 26 \end{bmatrix} = \begin{bmatrix} a+98 \\ 5a+22 \\ a^2+20a-26 \end{bmatrix}.$$

7 We have $\mathbf{A}^3 - \mathbf{A} = -\mathbf{I}$, so that $\mathbf{A}(\mathbf{A}^2 - \mathbf{I}) = -\mathbf{I}$, and hence $\mathbf{A}(-\mathbf{A}^2 + \mathbf{I}) = \mathbf{I}$. Similarly $(-\mathbf{A}^2 + \mathbf{I})\mathbf{A} = \mathbf{I}$. This shows that $-\mathbf{A}^2 + \mathbf{I}$ is an inverse for $\mathbf{A}$, and therefore $\mathbf{A}$ is invertible.

8a Since $\mathbf{A}$ is similar to $\mathbf{B}$ there exists invertible $\mathbf{T}$ such that $\mathbf{B} = \mathbf{TAT}^{-1}$. Now, $\mathbf{T}^{-1}$ is an invertible matrix such that

$$\mathbf{T}^{-1}\mathbf{BT} = \mathbf{T}^{-1}(\mathbf{TAT}^{-1})\mathbf{T} = (\mathbf{T}^{-1}\mathbf{T})\mathbf{A}(\mathbf{T}^{-1}\mathbf{T}) = \mathbf{IAI} = \mathbf{A},$$

and therefore $\mathbf{B}$ is similar to $\mathbf{A}$.

8b Suppose $\mathbf{A}$ is invertible, which is to say $\mathbf{A}^{-1}$ exists. Now, since $\mathbf{TT}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$ and $\mathbf{AA}^{-1} = \mathbf{I}$,

$$\mathbf{B}(\mathbf{TA}^{-1}\mathbf{T}^{-1}) = (\mathbf{TAT}^{-1})(\mathbf{TA}^{-1}\mathbf{T}^{-1}) = \mathbf{TA}(\mathbf{T}^{-1}\mathbf{T})\mathbf{A}^{-1}\mathbf{T}^{-1} = \mathbf{TAA}^{-1}\mathbf{T}^{-1} = \mathbf{TT}^{-1} = \mathbf{I},$$

and similarly $(\mathbf{TA}^{-1}\mathbf{T}^{-1})\mathbf{B} = \mathbf{I}$. This shows that $\mathbf{TA}^{-1}\mathbf{T}^{-1}$ is the inverse for $\mathbf{B}$, and therefore $\mathbf{B}$ is invertible.

If we next suppose that $\mathbf{B}$ is invertible, then since $\mathbf{B}$ is similar to $\mathbf{A}$ by part (a), a symmetrical argument (i.e. one in which we interchange the roles of $\mathbf{A}$ and $\mathbf{B}$ in the previous paragraph) shows that $\mathbf{A}$ must be invertible.

9 Performing row operations on

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

until $\mathbf{I}_3$ is obtained on the left side (the chosen series of steps can vary), we find that

$$\mathbf{C}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

10 The corresponding augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right].$$

We transform this matrix into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{array}\right] \xrightarrow[-3r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & -11 & -27 \end{array}\right] \xrightarrow{r_2\leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 9 \\ 0 & 0 & -7 & -17 \\ 0 & 0 & 0 & -2/7 \end{array}\right].$$

The third equation now states that $0 = -2/7$, which is a contradiction. Therefore the system has no solution.

11 Adding the equations gives $4x + 4z = 3$, so $z = \frac{3}{4} - x$. Putting this into the 1st equation yields $y = \frac{1}{2} - 3x$. Ergo the solution set is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : y = \tfrac{1}{2} - 3x \ \& \ z = \tfrac{3}{4} - x \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} t \\ 1/2 - 3t \\ 3/4 - \ \ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

12 If $[x, y, z]$ is a solution, then $c[x, y, z]$ is also a solution since $c[x, y, z] = [cx, cy, cz]$, and $x - 7y + 3z = 0$ implies $cx - 7(cy) + 3(cz) = 0$.

If $[x_1, y_1, z_1]$ and $[x_2, y_2, z_2]$ are solutions, then so too is

$$[x_1, y_1, z_1] + [x_2, y_2, z_2] = [x_1 + x_2, y_1 + y_2, z_1 + z_2]$$

since

$$(x_1 + x_2) - 7(y_1 + y_2) + 3(z_1 + z_2) = (x_1 - 7y_1 + 3z_1) + (x_2 - 7y_2 + 3z_2) = 0 + 0 = 0.$$

Observing that $[0, 0, 0]$ is a solution (so the solution set is nonempty), we conclude that the solution set is a subspace of $\mathbb{R}^3$.

Now, since $x = 7y - 3z$, the solution set is

$$\left\{ \begin{bmatrix} 7y - 3z \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} t : s, t \in \mathbb{R} \right\},$$

and therefore a basis is

$$\left\{ \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

13 $(\pi, 0)$ is a solution to $\sin x - 2y = 0$, but $\frac{1}{2}(\pi, 0) = (\pi/2, 0)$ is not: $\sin(\pi/2) - 2(0) = 1 \neq 0$. Not closed under scalar multiplication, and so not a subspace.

14 Since $ad - bc \neq 0$, either $a \neq 0$ or $c \neq 0$. By relabeling the coordinates of our two vectors if necessary, we can assume $a \neq 0$. Now, suppose that $x_1, x_2 \in \mathbb{R}$ are such that

$$x_1[a, b] + x_2[c, d] = [0, 0].$$

This gives the system

$$\begin{cases} x_1 a + x_2 c = 0 \\ x_1 b + x_2 d = 0 \end{cases}$$

From the 1st equation comes $x_1 = -(c/a)x_2$. Putting this into the 2nd equation gives

$$-\frac{bc}{a}x_2 + dx_2 = 0 \quad \Rightarrow \quad x_2\left(\frac{ad - bc}{a}\right) = 0 \quad \Rightarrow \quad x_2 = 0,$$

the last equation following from the middle one since $ad - bc \neq 0$. Now $x_1 = -(c/a)x_2 = 0$ as well, and we conclude that $[a, b]$ and $[c, d]$ are linearly independent.

15 Suppose

$$x_1[1, 2, 0] + x_2[1, 3, -1] + x_3[-1, 1, 1] = [0, 0, 0].$$

Then we obtain the system

$$\begin{cases} x_1 + x_2 - x_3 = 0 \\ 2x_1 + 3x_2 + x_3 = 0 \\ -x_2 + x_3 = 0 \end{cases}$$

The last equation gives $x_3 = x_2$, which can be used to go on to find that $x_1 = x_2 = x_3 = 0$. Therefore the vectors are linearly independent.