Math 260 Exam #3 Key (Summer 2017)

1 Complete the square:

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3x^2 - 2xy + 2y^2 = 3\left(x - \tfrac{1}{3}y\right)^2 + \tfrac{5}{3}y^2.$$

The expression at right can be seen to be never negative, and in order for it to equal zero it's necessary to have $x = \frac{1}{3}y$ and $y = 0$; that is, $x = y = 0$ is necessary. Thus $\mathbf{x}^\top \mathbf{A}\mathbf{x} \leq \mathbf{0}$ only if $\mathbf{x} = \mathbf{0}$, and therefore $\mathbf{A}$ is positive definite.

2 With the Gram-Schmidt Orthogonalization Process we obtain the orthonormal basis

$$\left(\frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{5\sqrt{3}} \begin{bmatrix} -1 \\ 7 \\ -5 \end{bmatrix} \right).$$

3 Let $\mathbf{w}_1 = 1$, $\mathbf{u}_2 = t$, and $\mathbf{u}_3 = t^2$. Then by the Gram-Schmidt procedure,

$$\mathbf{w}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 = t - \frac{\int_0^1 t\,dt}{\int_0^1 1\,dt}(1) = t - \frac{1}{2},$$

and

$$\mathbf{w}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{w}_1 \rangle}{\langle \mathbf{w}_1, \mathbf{w}_1 \rangle} \mathbf{w}_1 - \frac{\langle \mathbf{u}_3, \mathbf{w}_2 \rangle}{\langle \mathbf{w}_2, \mathbf{w}_2 \rangle} \mathbf{w}_2 = t^2 - \frac{\int_0^1 t^2\,dt}{\int_0^1 1\,dt}(1) - \frac{\int_0^1 t^2\left(t - \frac{1}{2}\right)dt}{\int_0^1 \left(t - \frac{1}{2}\right)^2 dt}\left(t - \tfrac{1}{2}\right) = t^2 - t + \tfrac{1}{6}.$$

So $\left\{1, t - \frac{1}{2}, t^2 - t + \frac{1}{6}\right\}$ is an orthogonal basis, and it remains to normalize the basis elements. Let $\hat{\mathbf{w}}_1 = \mathbf{w}_1$,

$$\hat{\mathbf{w}}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{t - \frac{1}{2}}{\sqrt{\int_0^1 \left(t - \frac{1}{2}\right)^2 dt}} = \frac{t - \frac{1}{2}}{\sqrt{\frac{1}{12}}} = (2t - 1)\sqrt{3},$$

and

$$\hat{\mathbf{w}}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{t^2 - t + \frac{1}{6}}{\sqrt{\int_0^1 \left(t^2 - t + \frac{1}{6}\right)^2 dt}} = \frac{t^2 - t + \frac{1}{6}}{\sqrt{\frac{1}{180}}} = (6t^2 - 6t + 1)\sqrt{5},$$

Then $\{\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3\}$ is an orthonormal basis.

4 The matrix is upper-triangular, and so the determinant is simply the product of the diagonal entries: $4a^2$.

5 Perform the column operations $c_1 + c_3 \to c_3$ and $-c_1 + c_4 \to c_4$ to obtain

$$\begin{vmatrix} -2 & 2 & 1 & -2 \\ 1 & 1 & 7 & 2 \\ -1 & 0 & 0 & 0 \\ -2 & t & 4 & -3 \end{vmatrix} = -\begin{vmatrix} 2 & 1 & -2 \\ 1 & 7 & 2 \\ t & 4 & -3 \end{vmatrix}.$$

Next, it's convenient to perform the row operations $r_1 \leftrightarrow r_2$ and $-2r_1 + r_2 \to r_2$ to obtain

$$-\begin{vmatrix} 2 & 1 & -2 \\ 1 & 7 & 2 \\ t & 4 & -3 \end{vmatrix} = \begin{vmatrix} 1 & 7 & 2 \\ 0 & -13 & -4 \\ t & 4 & -3 \end{vmatrix} = \begin{vmatrix} -13 & -4 \\ 4 & -3 \end{vmatrix} + t\begin{vmatrix} 7 & 2 \\ -13 & -4 \end{vmatrix} = 55 - 2t.$$

6 Performing the column operation $c_3 + c_2 \to c_2$ and then expanding along the 3rd row gives

$$\det(\mathbf{A}) = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = -6.$$

Now, with $\mathbf{A} = [\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3]$, we have

$$x = \det(\mathbf{b} \ \mathbf{a}_2 \ \mathbf{a}_3) = \begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 2, \quad y = \det(\mathbf{a}_1 \ \mathbf{b} \ \mathbf{a}_3) = \begin{vmatrix} 3 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -4,$$

and

$$z = \det(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{b}) = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2.$$

By Cramer's Rule the solution to the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\frac{\det(\mathbf{b} \ \mathbf{a}_2 \ \mathbf{a}_3)}{\det(\mathbf{A})}, \ \frac{\det(\mathbf{a}_1 \ \mathbf{b} \ \mathbf{a}_3)}{\det(\mathbf{A})}, \ \frac{\det(\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{b})}{\det(\mathbf{A})}\right]^\top = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$

7a We have

$$P_{\mathbf{B}}(t) = \det(\mathbf{B} - t\mathbf{I}) = \begin{vmatrix} 1-t & -3 & 3 \\ 3 & -5-t & 3 \\ 6 & -6 & 4-t \end{vmatrix} = -(t+2)^2(4-t),$$

and so the characteristic equation for $\mathbf{B}$ is $(t+2)^2(4-t) = 0$. The eigenvalues are the roots of the characteristic equation: -2 and 4.

7b The eigenspace corresponding to -2 is the solution space of the system $\mathbf{Bx} = -2\mathbf{x}$, or equivalently $(\mathbf{B} + 2\mathbf{I})\mathbf{x} = \mathbf{0}$, which may be written as

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus

$$E_{\mathbf{B}}(-2) = \left\{ s\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\},$$

which has basis $\{[1, 0, -1]^\top, \ [0, 1, 1]^\top\}$.

The eigenspace corresponding to 4 is the solution space of the system $\mathbf{B}\mathbf{x} = 4\mathbf{x}$, or equivalently $(\mathbf{B} - 4\mathbf{I})\mathbf{x} = \mathbf{0}$, which may be written as

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus

$$E_{\mathbf{B}}(4) = \left\{ t\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\},$$

which has basis $\left\{[1,1,2]^\top\right\}$.

8a The characteristic polynomial for $\mathbf{N}$ is

$$P_{\mathbf{N}}(t) = \det(\mathbf{N} - t\mathbf{I}) = \begin{vmatrix} -2-t & 8 \\ -1 & 4-t \end{vmatrix} = (t+2)(t-4) + 8 = t(t-2),$$

and so the eigenvalues are 0 and 2.

8b Eigenspace for $\lambda = 0$:

$$\begin{aligned} E_{\mathbf{N}}(0) = \{\mathbf{x} : \mathbf{N}\mathbf{x} = \mathbf{0}\} &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} -2 & 8 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = 4y \right\} = \left\{ t\begin{bmatrix} 4 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} 4 \\ 1 \end{bmatrix} \right), \end{aligned}$$

and so $\left([4,1]^\top\right)$ is a basis for $E_{\mathbf{N}}(0)$.

Eigenspace for $\lambda = 2$:

$$\begin{aligned} E_{\mathbf{N}}(2) = \{\mathbf{x} : \mathbf{N}\mathbf{x} = 2\mathbf{x}\} &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : \begin{bmatrix} -2 & 8 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} \right\} \\ &= \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x = 2y \right\} = \left\{ t\begin{bmatrix} 2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right), \end{aligned}$$

and so $\left([2,1]^\top\right)$ is a basis for $E_{\mathbf{N}}(2)$.

8c By the diagonalization procedure,

$$\mathbf{P} = \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{P}^{-1} = \tfrac{1}{2}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix}$$

8d We have

$$\mathbf{N}^{10} = (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})^{10} = \mathbf{P}\mathbf{D}^{10}\mathbf{P}^{-1} = \tfrac{1}{2}\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 2^{10} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} -2^{10} & 2^{12} \\ -2^{9} & 2^{11} \end{bmatrix}.$$

8e We have

$$\mathbf{N}^{1/2} = (\mathbf{PDP}^{-1})^{1/2} = \mathbf{PD}^{1/2}\mathbf{P}^{-1} = \tfrac{1}{2}\begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ -1 & 4 \end{bmatrix} = \sqrt{2}\begin{bmatrix} -1 & 4 \\ -1/2 & 2 \end{bmatrix}.$$