Math 260 Exam #2 Key (Summer 2017)

1 Let $\mathbf{p}_1, \mathbf{p}_2 \in aS + \mathbf{b}$, so $\mathbf{p}_1 = a\mathbf{s}_1 + \mathbf{b}$ and $\mathbf{p}_2 = a\mathbf{s}_2 + \mathbf{b}$ for some $\mathbf{s}_1, \mathbf{s}_2 \in S$. Let $\mathbf{x} \in [\mathbf{p}_1, \mathbf{p}_2]$, so $\mathbf{x} = (1-t)\mathbf{p}_1 + t\mathbf{p}_2$ for some $t \in [0,1]$. Now, since $(1-t)\mathbf{s}_1 + t\mathbf{s}_2 \in S$ on account of S being convex, we find that

$$\mathbf{x} = (1-t)(a\mathbf{s}_1 + \mathbf{b}) + t(a\mathbf{s}_2 + \mathbf{b}) = a[(1-t)\mathbf{s}_1 + t\mathbf{s}_2] + \mathbf{b} \in aS + \mathbf{b}.$$

Thus $[\mathbf{p}_1, \mathbf{p}_2] \subseteq aS + \mathbf{b}$, and therefore $aS + \mathbf{b}$ is convex.

2 Possible dimensions of V are 0, 1, 2. If $V \neq \mathbb{R}^2$ then $\dim V$ is either 0 or 1. If $\dim V = 0$ then $V = \{\mathbf{0}\}$. If $\dim V = 1$, then V has basis $\{\mathbf{b}\}$ for some nonzero vector $\mathbf{b} \in \mathbb{R}^2$, so that $V = \operatorname{Span}(\mathbf{b}) = \{t\mathbf{b} : t \in \mathbb{R}\}$, which is a line through the origin.

3 In terms of column vectors we have $\mathbf{A} = [\mathbf{a}_1 \ \cdots \ \mathbf{a}_n]$ and $\mathbf{B} = [\mathbf{b}_1 \ \cdots \ \mathbf{b}_r]$. Now, since

$$\mathbf{AB} = [\mathbf{Ab}_1 \ \cdots \ \mathbf{Ab}_r],$$

the ℓth column vector of $\mathbf{AB}$ is $\mathbf{Ab}_\ell$. Set $\mathbf{b}_\ell = [b_{1\ell} \ \cdots \ b_{n\ell}]^\top$. Then, letting a_{ij} denote the ij-entry of $\mathbf{A}$ in general,

$$\mathbf{Ab}_\ell = \begin{bmatrix} \sum_{j=1}^n a_{1j}b_{j\ell} \\ \vdots \\ \sum_{j=1}^n a_{mj}b_{j\ell} \end{bmatrix} = \sum_{j=1}^n \begin{bmatrix} a_{1j}b_{j\ell} \\ \vdots \\ a_{mj}b_{j\ell} \end{bmatrix} = \sum_{j=1}^n b_{j\ell} \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix} = \sum_{j=1} b_{j\ell}\mathbf{a}_j.$$

Thus each of the column vectors of $\mathbf{AB}$ is a linear combination of the column vectors of $\mathbf{A}$, so that the column space of $\mathbf{AB}$ is a subset of the column space of $\mathbf{A}$: $\operatorname{Col}(\mathbf{AB}) \subseteq \operatorname{Col}(\mathbf{A})$. Therefore

$$\operatorname{rank}(\mathbf{AB}) = \dim[\operatorname{Col}(\mathbf{AB})] \leq \dim[\operatorname{Col}(\mathbf{A})] = \operatorname{rank}(\mathbf{A}).$$

4 With the elementary row operations $-3r_1 + r_2$, $r_1 + r_3$, and $-3r_1 + r_4$ we obtain

$$\mathbf{M} \sim \begin{bmatrix} 1 & -2 & 0 & 4 \\ 0 & 7 & 1 & -12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The rank of $\mathbf{M}$ is equal to the number of pivots in a row-equivalent row-echelon form, and so $\operatorname{rank}(\mathbf{M}) = 2$.

5 The image of the line $x = c$ under F is $\{F(c,y) : y \in \mathbb{R}\}$, where

$$F(c,y) = (e^c \sin y, e^c \cos y).$$

Letting $u = e^c \sin y$ and $v = e^c \cos y$, we have $u^2 + v^2 = e^c$. Thus the image of $x = c$ under F is a circle with center $(0,0)$ and radius e^c.

6 By linearity $L(1,-1) = -L(-1,1) = (-6,-3)$, so

$$L(2,0) = L((1,1) + (1,-1)) = L(1,1) + L(1,-1) = (2,1) + (-6,-3) = (-4,-2),$$

and thus $L(1,0) = \frac{1}{2}L(2,0) = (-2,-1)$.

7 Suppose $\sum_{k=1}^{n} c_k \mathbf{v}_k = \mathbf{0}$. Then

$$\mathbf{0} = L(\mathbf{0}) = L\left(\sum_{k=1}^{n} c_k \mathbf{v}_k\right) = \sum_{k=1}^{n} c_k L(\mathbf{v}_k) = \sum_{k=1}^{n} c_k \mathbf{w}_k$$

by linearity properties, and since $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent we conclude that $c_k = 0$ for all k.

8 By the Rank-Nullity Theorem, $\dim(\operatorname{Img} L) + \dim(\operatorname{Ker} L) = \dim V$. Since $\operatorname{Img} L$ is a subspace of W, $\dim(\operatorname{Img} L) < \dim W$. Thus $\dim(\operatorname{Img} L) < \dim V$, and it follows that $\dim(\operatorname{Ker} L) > 0$. That is, $\dim(\operatorname{Ker} L) \geq 1$, and we conclude that $\operatorname{Ker} L \neq \{\mathbf{0}\}$.

9 Let

$$\mathbf{v}_1^\top = [1, 1, -2, 3, 4, 5] \quad \text{and} \quad \mathbf{v}_2^\top = [1, 0, 0, 2, 0, 8],$$

and let $\mathbf{A}$ be the 2×6 matrix with row vectors $\mathbf{v}_1^\top$ and $\mathbf{v}_2^\top$. It is easy to show that two column vectors of $\mathbf{A}$ are linearly independent, so $\operatorname{rank} \mathbf{A} = 2$. Let U be the subspace in question, so

$$U = \{\mathbf{x} \in \mathbb{R}^6 : \mathbf{v}_1 \cdot \mathbf{x} = 0 \text{ and } \mathbf{v}_2 \cdot \mathbf{x} = 0\} = \{\mathbf{x} \in \mathbb{R}^6 : \mathbf{A}\mathbf{x} = \mathbf{0}\}.$$

By the Rank-Nullity Theorem adapted for matrices we have

$$\dim U = \dim(\operatorname{Nul} \mathbf{A}) = \operatorname{nullity} \mathbf{A} = \dim \mathbb{R}^6 - \operatorname{rank} \mathbf{A} = 6 - 2 = 4.$$

10 By inspection,

$$[L] = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}.$$

11 Let $\mathbf{v} \in V$. Now, $(P \circ P)(\mathbf{v}) = P(\mathbf{v})$ shows that $P(\mathbf{v}) \in \operatorname{Img} P$. Also

$$P(\mathbf{v} - P(\mathbf{v})) = P(\mathbf{v}) - P(P(\mathbf{v})) = P(\mathbf{v}) - P(\mathbf{v}) = \mathbf{0},$$

which shows $\mathbf{v} - P(\mathbf{v}) \in \operatorname{Ker} P$. Thus

$$\mathbf{v} = [\mathbf{v} - P(\mathbf{v})] + P(\mathbf{v}) \in \operatorname{Ker} P + \operatorname{Img} P,$$

so that $V \subseteq \operatorname{Ker} P + \operatorname{Img} P$. That $\operatorname{Ker} P + \operatorname{Img} P \subseteq V$ follows from the usual closure properties of a vector space, and therefore $V = \operatorname{Ker} P + \operatorname{Img} P$.

12 Suppose $L(x, y) = (0, 0)$. Then $2x + y = 0$ and $3x - 5y = 0$, which are only satisfied if $x = y = 0$. Thus $\operatorname{Ker} L = \{(0, 0)\}$, which implies L is injective, hence surjective, hence bijective, and therefore invertible.

13 Setting $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$ and $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2)$, the transition matrix is $\mathbf{I}_{\mathcal{BC}} = \big[[\mathbf{v}_1]_{\mathcal{C}} \quad [\mathbf{v}_2]_{\mathcal{C}}\big]$. Here $[\mathbf{v}_1]_{\mathcal{C}} = [a\ b]^\top$ for a, b such that $a\mathbf{w}_1 + b\mathbf{w}_2 = \mathbf{v}_1$, and $[\mathbf{v}_2]_{\mathcal{C}} = [c\ d]^\top$ for c, d such that $c\mathbf{w}_1 + d\mathbf{w}_2 = \mathbf{v}_2$. Solve these two simple systems of equations yields $a = -3/5$, $b = 2/5$, $c = 3$, and $d = 0$, and so

$$\mathbf{I}_{\mathcal{BC}} = \begin{bmatrix} -3/5 & 3 \\ 2/5 & 0 \end{bmatrix}.$$