Math 260 Exam #1 Key (Summer 2017)

1a $\|\mathbf{u}\| = \sqrt{(-3)^2 + 9^2 + 2^2} = \sqrt{94}$ and $\|\mathbf{v}\| = \sqrt{(-2)^2 + (-1)^2 + 0^2} = \sqrt{5}$.

1b Since $\mathbf{u} \cdot \mathbf{v} = (-3)(-2) + (9)(-1) + (2)(0) = -3$ and $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = 5$,

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} = -\tfrac{3}{5}[-2, -1, 0] = \left[\tfrac{6}{5}, \tfrac{3}{5}, 0\right].$$

1c We have

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = -\frac{3}{\sqrt{94}\sqrt{5}} = -\frac{3}{\sqrt{470}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(-\frac{3}{\sqrt{470}}\right) \approx 97.95^\circ.$$

2 The constant is 4:

$$\begin{aligned}
\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}-\mathbf{v}\|^2 &= (\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) - (\mathbf{u}-\mathbf{v}) \cdot (\mathbf{u}-\mathbf{v}) \\
&= (\mathbf{u}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}) - (\mathbf{u}\cdot\mathbf{u} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{v}\cdot\mathbf{v}) \\
&= 2\mathbf{u}\cdot\mathbf{v} + 2\mathbf{u}\cdot\mathbf{v} = 4\mathbf{u}\cdot\mathbf{v}.
\end{aligned}$$

3 A parametrization of the segment is

$$\mathbf{x}(t) = (1-t)\mathbf{p} + t\mathbf{q} = [1-5t, 3-t, -1+7t],$$

and the point in question is $\mathbf{x}(1/5) = [0, 14/5, 2/5]$.

4 The plane consists of all points x such that $\mathbf{n} \cdot \overrightarrow{px} = 0$, giving $\mathbf{n} \cdot (\mathbf{x} - \mathbf{p}) = 0$, so that

$$[-3, 2, 8] \cdot [x - 1/2, y, z - 2] = 0,$$

and finally

$$-3x + 2y + 8z = \tfrac{29}{2}.$$

5 Two points determine the line, and they can be found as solutions to the system

$$\begin{cases} x - y + z = 2 \\ 2x - 3y + z = 1 \end{cases}$$

The first equation gives $z = 2 - x + y$, which when put into the second equation gives $x = 2y - 1$. Putting this back into $z = 2 - x + y$ gives $z = 3 - y$. The solution set of the system is

$$S = \big\{[2y-1, y, 3-y] : y \in \mathbb{R}\big\}.$$

Replacing y with t:

$$S = \big\{[-1, 0, 3] + t[2, 1, -1] : t \in \mathbb{R}\big\}.$$

Thus

$$\mathbf{x}(t) = [-1, 0, 3] + t[2, 1, -1]$$

is a parametric equation for the line of intersection of the two planes.

6 We have

$$\mathbf{ABC} = \mathbf{A}(\mathbf{BC}) = \mathbf{A}\begin{bmatrix} 4 \\ -7 \\ a-3 \end{bmatrix} = \begin{bmatrix} -6a+5 \\ 2a+13 \end{bmatrix}.$$

7 We have $\mathbf{A}^2 + 2\mathbf{A} = -\mathbf{I}$, so that $\mathbf{A}(\mathbf{A}+2\mathbf{I}) = -\mathbf{I}$, and hence $\mathbf{A}(-\mathbf{A}-2\mathbf{I}) = \mathbf{I}$. Similarly $(-\mathbf{A}-2\mathbf{I})\mathbf{A} = \mathbf{I}$. This shows that $-\mathbf{A}-2\mathbf{I}$ is an inverse for $\mathbf{A}$, and therefore $\mathbf{A}$ is invertible.

8 Performing row operations on

$$\left[\begin{array}{ccc|ccc} 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 3 & 0 & 1 & 0 \\ 1 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

until $\mathbf{I}_3$ is obtained on the left side (the chosen series of steps can vary), we find that

$$\mathbf{C}^{-1} = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

9 The corresponding augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 2 & 0 & -1 & -2 \\ 3 & 5 & 2 & 22 \end{array}\right].$$

We transform this matrix into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 2 & 0 & -1 & -2 \\ 3 & 5 & 2 & 22 \end{array}\right] \xrightarrow[-3r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -4 & 1 & -20 \\ 0 & -1 & 5 & -5 \end{array}\right] \xrightarrow{r_2\leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -1 & 5 & -5 \\ 0 & -4 & 1 & -20 \end{array}\right]$$

$$\xrightarrow{-4r_2+r_3\to r_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -1 & 5 & -5 \\ 0 & 0 & -19 & 0 \end{array}\right].$$

We have obtained the equivalent system of equations

$$\begin{cases} x + 2y - z = 9 \\ - y + 5z = -5 \\ - 19z = 0 \end{cases}$$

From the third equation we have $z = 0$, which when put into the second equation yields $-y = -5$, or $y = 5$. Finally, from the first equation we obtain.

$$x + 2(5) - 0 = 9 \;\Rightarrow\; x = -1.$$

Therefore the sole solution to the system is $(-1, 5, 0)$.

10 From 2nd equation: $x = 2z - y$. Put this into 1st equation:

$$3(2z - y) - y + 6z = 4 \quad \Rightarrow \quad y = 3z - 1.$$

Now $x = 2z - y - 1$ gives

$$x = 2z - (3z - 1) = 1 - z.$$

Solution set is therefore

$$\left\{ \begin{bmatrix} 1-z \\ 3z-1 \\ z \end{bmatrix} : z \in \mathbb{R} \right\}.$$

11 The set S is not closed under scalar multiplication. For instance, $[0,3]^\top \in S$, but $2[0,3]^\top = [0,6]^\top \notin S$ since $0^2 + 6^2 > 9$. Thus S is not a vector space.

12 We have $\mathbf{u} = [4,3] \in U$ and $\mathbf{v} = [-1,8] \in V$, so that $\mathbf{u}, \mathbf{v} \in U \cup V$. Now, $\mathbf{u} + \mathbf{v} = [3,11]$, but $4(11) - 3(3) \neq 0$ and $11 + 8(3) \neq 0$ show that $\mathbf{u} + \mathbf{v} \notin U$ and $\mathbf{u} + \mathbf{v} \notin V$, respectively. Hence $\mathbf{u} + \mathbf{v} \notin U + V$, which shows that $U + V$ is not closed under addition. Therefore $U \cup V$ is not a vector subspace of $\mathbb{R}^2$.

13 Suppose $\mathbf{x}, \mathbf{y} \in U_1 + U_2$ and c is a scalar. Thus there exist $\mathbf{x}_1, \mathbf{y}_1 \in U_1$ and $\mathbf{x}_2, \mathbf{y}_2 \in U_2$ such that $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{y} = \mathbf{y}_1 + \mathbf{y}_2$. Now,

$$\mathbf{x} + \mathbf{y} = (\mathbf{x}_1 + \mathbf{x}_2) + (\mathbf{y}_1 + \mathbf{y}_2) = (\mathbf{x}_1 + \mathbf{y}_1) + (\mathbf{x}_2 + \mathbf{y}_2),$$

and since $\mathbf{x}_1 + \mathbf{y}_1 \in U_1$ and $\mathbf{x}_2 + \mathbf{y}_2 \in U_2$, it follows that $\mathbf{x} + \mathbf{y} \in U_1 + U_2$.

Next, we also have $c\mathbf{x}_1 \in U_1$ and $c\mathbf{x}_2 \in U_2$, and since $c\mathbf{x} = c\mathbf{x}_1 + c\mathbf{x}_2$, we conclude that $c\mathbf{x} \in U_1 + U_2$.

Finally, since $\mathbf{0} = \mathbf{0} + \mathbf{0}$ with $\mathbf{0} \in U_1$ and $\mathbf{0} \in U_2$, we see that $\mathbf{0} \in U_1 + U_2$ and hence $U_1 + U_2 \neq \varnothing$. Therefore $U_1 + U_2$ is a subspace of V.

14 Suppose that $c_1[1,2] + c_2[1,3] = [0,0]$. This yields the system $c_1 + c_2 = 0$, $2c_1 + 3c_2 = 0$, which gives $c_1 = c_2 = 0$ as the only solution. Therefore the vectors are linear independent.