Math 260 Exam #3 Key (Summer 2016)

1a Let $\mathbf{x} = [x, y]^\top \in \mathbb{R}^2$. Completing the square gives

$$\begin{aligned}\mathbf{x}^\top\mathbf{A}\mathbf{x} &= \begin{bmatrix} x & y\end{bmatrix}\begin{bmatrix} 4 & -1 \\ -1 & 10\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = 4x^2 - 2xy + 10y^2 = 4\big(x^2 - \tfrac{1}{2}xy\big) + 10y^2 \\ &= 4\big[x^2 + \big(-\tfrac{1}{2}y\big)x + \tfrac{1}{16}y^2\big] - 4\big(\tfrac{1}{16}y^2\big) + 10y^2 = 4\big(x - \tfrac{1}{4}y\big)^2 + \tfrac{39}{4}y^2 \\ &= \big(2x - \tfrac{1}{2}y\big)^2 + \tfrac{39}{4}y^2,\end{aligned}$$

a sum of squares, which cannot equal zero for $x, y \in \mathbb{R}$ unless $x = y = 0$. That is $\mathbf{x}^\top\mathbf{A}\mathbf{x} = 0$ if and only if $\mathbf{x} = \mathbf{0}$, and therefore $\mathbf{A}$ is positive definite.

1b Suppose $ac - b^2 \le 0$. Then in particular it is possible that $ac - b^2 = 0$, giving $b^2 = ac$, and hence $c \ge 0$ since $a > 0$ and b^2 cannot be negative. Now,

$$\mathbf{x}^\top\mathbf{A}\mathbf{x} = \begin{bmatrix} x & y\end{bmatrix}\begin{bmatrix} a & b \\ b & c\end{bmatrix}\begin{bmatrix} x \\ y\end{bmatrix} = ax^2 + 2bxy + cy^2,$$

and so if we let $\mathbf{x} = \big[\sqrt{c}, -\sqrt{a}\,\big]^\top$, then $\mathbf{x} \in \mathbb{R}^2$ is a nonzero vector such that $\mathbf{x}^\top\mathbf{A}\mathbf{x} = \mathbf{0}$, and we conclude that $\mathbf{A}$ is not positive definite. Therefore if $\mathbf{A}$ is positive definite, then is must be that $ac - b^2 > 0$.

For the converse, suppose $\mathbf{A}$ is not positive definite. Then there exists some $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{x} \ne \mathbf{0}$ and $\mathbf{x}^\top\mathbf{A}\mathbf{x} \le 0$. Now,

$$\begin{aligned}\mathbf{x}^\top\mathbf{A}\mathbf{x} \le 0 \;\Rightarrow\; & ax^2 + 2bxy + cy^2 \le 0 \;\Rightarrow\; x^2 + \frac{2by}{a}x + \frac{c}{a}y^2 \le 0 \\ \Rightarrow\; & \left[x^2 + \frac{2by}{a}x + \left(\frac{by}{a}\right)^2\right] - \left(\frac{by}{a}\right)^2 + \frac{c}{a}y^2 \le 0 \\ \Rightarrow\; & \left(x + \frac{b}{a}y\right)^2 - \left(\frac{b^2 - ac}{a^2}\right)y^2 \le 0.\end{aligned}$$

To satisfy the last inequality requires that

$$\frac{b^2 - ac}{a^2} \ge 0,$$

which in turn requires that $b^2 - ac \ge 0$, and hence $ac - b^2 \le 0$. Therefore if $ac - b^2 > 0$, then $\mathbf{A}$ must be positive definite.

2 Let $\mathbf{w}_1 = 1$, $\mathbf{u}_2 = t$, and $\mathbf{u}_3 = t^2$. Then by the Gram-Schmidt procedure,

$$\mathbf{w}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 = t - \frac{\int_0^1 t\,dt}{\int_0^1 1\,dt}(1) = t - \frac{1}{2},$$

and

$$\mathbf{w}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{w}_1\rangle}{\langle \mathbf{w}_1, \mathbf{w}_1\rangle}\mathbf{w}_1 - \frac{\langle \mathbf{u}_3, \mathbf{w}_2\rangle}{\langle \mathbf{w}_2, \mathbf{w}_2\rangle}\mathbf{w}_2 = t^2 - \frac{\int_0^1 t^2\,dt}{\int_0^1 1\,dt}(1) - \frac{\int_0^1 t^2\big(t - \frac{1}{2}\big)\,dt}{\int_0^1 \big(t - \frac{1}{2}\big)^2 dt}\big(t - \tfrac{1}{2}\big) = t^2 - t + \tfrac{1}{6}.$$

So $\left\{1, t-\frac{1}{2}, t^2-t+\frac{1}{6}\right\}$ is an orthogonal basis, and it remains to normalize the basis elements. Let $\hat{\mathbf{w}}_1 = \mathbf{w}_1$,

$$\hat{\mathbf{w}}_2 = \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|} = \frac{t-\frac{1}{2}}{\sqrt{\int_0^1 \left(t-\frac{1}{2}\right)^2 dt}} = \frac{t-\frac{1}{2}}{\sqrt{\frac{1}{12}}} = (2t-1)\sqrt{3},$$

and

$$\hat{\mathbf{w}}_3 = \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} = \frac{t^2-t+\frac{1}{6}}{\sqrt{\int_0^1 \left(t^2-t+\frac{1}{6}\right)^2 dt}} = \frac{t^2-t+\frac{1}{6}}{\sqrt{\frac{1}{180}}} = (6t^2-6t+1)\sqrt{5},$$

Then $\{\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2, \hat{\mathbf{w}}_3\}$ is an orthonormal basis.

3 Perform the row operations $-2r_3 + r_1 \to r_1$, $-2r_3 + r_4 \to r_4$, and $r_3 + r_2 \to r_2$ to get

$$\begin{vmatrix} 0 & 2 & 1 & -2 \\ 0 & 1 & t+1 & 2 \\ -1 & 0 & 1 & -1 \\ 0 & 2 & 4 & -3 \end{vmatrix} = -\begin{vmatrix} 2 & 1 & -2 \\ 1 & t+1 & 2 \\ 2 & 4 & -3 \end{vmatrix}.$$

Now perform operations $-2r_2 + r_1 \to r_1$, $-2r_2 + r_3 \to r_3$:

$$-\begin{vmatrix} 0 & -2t-1 & -6 \\ 1 & t+1 & 2 \\ 0 & -2t+2 & -7 \end{vmatrix} = \begin{vmatrix} -2t-1 & -6 \\ -2t+2 & -7 \end{vmatrix} = 2t+19.$$

4 Performing the column operation $c_3 + c_2 \to c_2$ and then expanding along the 3rd row gives

$$\det(\mathbf{A}) = \begin{vmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 0 & -1 \\ 1 & 2 & 1 \\ 0 & 0 & -1 \end{vmatrix} = (-1)\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} = -6.$$

Now, with $\mathbf{A} = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3]$, we have

$$x = \det(\mathbf{b} \;\; \mathbf{a}_2 \;\; \mathbf{a}_3) = \begin{vmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 2, \quad y = \det(\mathbf{a}_1 \;\; \mathbf{b} \;\; \mathbf{a}_3) = \begin{vmatrix} 3 & 0 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{vmatrix} = -4,$$

and

$$z = \det(\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{b}) = \begin{vmatrix} 3 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2.$$

By Cramer's Rule the solution to the system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left[\frac{\det(\mathbf{b} \;\; \mathbf{a}_2 \;\; \mathbf{a}_3)}{\det(\mathbf{A})}, \frac{\det(\mathbf{a}_1 \;\; \mathbf{b} \;\; \mathbf{a}_3)}{\det(\mathbf{A})}, \frac{\det(\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{b})}{\det(\mathbf{A})}\right]^\top = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix}.$$

5a We have

$$P_{\mathbf{B}}(t) = \det(\mathbf{B} - t\mathbf{I}) = \begin{vmatrix} 1-t & -3 & 3 \\ 3 & -5-t & 3 \\ 6 & -6 & 4-t \end{vmatrix} = -(t+2)^2(4-t),$$

and so the characteristic equation for $\mathbf{B}$ is $(t+2)^2(4-t) = 0$.

5b The eigenvalues are the roots to the characteristic equation: -2 and 4.

5c The eigenspace corresponding to -2 is the solution space of the system $\mathbf{B}\mathbf{x} = -2\mathbf{x}$, or equivalently $(\mathbf{B} + 2\mathbf{I})\mathbf{x} = \mathbf{0}$, which may be written as

$$\begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus

$$E_{\mathbf{B}}(-2) = \left\{ s \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\},$$

which has basis $\left\{ [1, 0, -1]^\top,\ [0, 1, 1]^\top \right\}$.

The eigenspace corresponding to 4 is the solution space of the system $\mathbf{B}\mathbf{x} = 4\mathbf{x}$, or equivalently $(\mathbf{B} - 4\mathbf{I})\mathbf{x} = \mathbf{0}$, which may be written as

$$\begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus

$$E_{\mathbf{B}}(4) = \left\{ t \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\},$$

which has basis $\left\{ [1, 1, 2]^\top \right\}$.

6a We have

$$P_{\mathbf{M}}(t) = \det(\mathbf{M} - t\mathbf{I}) = \begin{vmatrix} 7-t & -15 \\ 2 & -4-t \end{vmatrix} = (t-1)(t-2),$$

and so the eigenvalues of $\mathbf{M}$ are 1 and 2.

6b A basis for $E_{\mathbf{M}}(1)$ is $\left\{ [5, 2]^\top \right\}$, and a basis for $E_{\mathbf{M}}(2)$ is $\left\{ [3, 1]^\top \right\}$.

6c With the appropriate theorem, we find that

$$\mathbf{P} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

6d After finding that

$$\mathbf{P}^{-1} = \begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix},$$

we obtain

$$\mathbf{M}^{12} = (\mathbf{PDP}^{-1})^{12} = \mathbf{PD}^{12}\mathbf{P}^{-1} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4096 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} 24{,}571 & -61{,}425 \\ 8{,}190 & -20{,}474 \end{bmatrix}.$$

6e We have

$$\mathbf{M}^{1/2} = (\mathbf{PDP}^{-1})^{1/2} = \mathbf{PD}^{1/2}\mathbf{P}^{-1} = \begin{bmatrix} 5 & 3 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 2 & -5 \end{bmatrix} = \begin{bmatrix} -5+6\sqrt{2} & 15-15\sqrt{2} \\ -2+2\sqrt{2} & 6-5\sqrt{2} \end{bmatrix}.$$