Math 260 Exam #2 Key (Summer 2016)

1 Let $\mathbf{x},\mathbf{y}\in S$. So $\mathbf{x}^\top\mathbf{a}\geq c$ and $\mathbf{y}^\top\mathbf{a}\geq c$. Suppose $\mathbf{z}\in\overline{\mathbf{xy}}$, so $\mathbf{z}=\mathbf{x}+t(\mathbf{y}-\mathbf{x})$ for some $t\in[0,1]$. Now,

$$\begin{aligned}\mathbf{z}^\top c &= [\mathbf{x}+t(\mathbf{y}-\mathbf{x})]^\top a = [\mathbf{x}^\top + t(\mathbf{y}^\top-\mathbf{x}^\top)]a = \mathbf{x}^\top a + t(\mathbf{y}^\top a - \mathbf{x}^\top a)\\ &= (1-t)\mathbf{x}^\top a + t\mathbf{y}^\top a \geq (1-t)c + tc = c,\end{aligned}$$

and hence $\mathbf{z}\in S$. We conclude that $\overline{\mathbf{xy}}\subseteq S$, and therefore S is a convex set.

2 Since

$$V=\left\{\begin{bmatrix}0&0&0\\a&0&0\\b&c&0\end{bmatrix}\,\middle|\, a,b,c\in\mathbb{R}\right\}=\operatorname{Span}\left\{\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\right\},$$

the set

$$\mathcal{B}=\left\{\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix}\right\}$$

is a basis for V, and therefore $\dim(V)=3$.

3 Find an equivalent matrix to $\mathbf{H}$ that is in row-echelon form:

$$\mathbf{H}\xrightarrow[r_2+r_3]{-r_1+r_4}\begin{bmatrix}-2&2&3&-4&-1\\1&1&-2&3&1\\0&4&-1&2&1\\0&0&3&0&0\end{bmatrix}\xrightarrow{r_1\leftrightarrow r_2}\begin{bmatrix}1&1&-2&3&1\\-2&2&3&-4&-1\\0&4&-1&2&1\\0&0&3&0&0\end{bmatrix}$$

$$\xrightarrow{2r_1+r_2}\begin{bmatrix}1&1&-2&3&1\\0&4&-1&2&1\\0&4&-1&2&1\\0&0&3&0&0\end{bmatrix}\xrightarrow{r_3\leftrightarrow r_4}\begin{bmatrix}1&1&-2&3&1\\0&4&-1&2&1\\0&0&3&0&0\\0&0&0&0&0\end{bmatrix}.$$

The row-echelon form has three nonzero row vectors, which implies that $\operatorname{rank}(\mathbf{H})=3$.

4 Behold, for T be nonlinear:

$$T\left(\begin{bmatrix}2\\0\end{bmatrix}+\begin{bmatrix}0\\2\end{bmatrix}\right)=T\left(\begin{bmatrix}2\\2\end{bmatrix}\right)=\begin{bmatrix}4\\2\end{bmatrix}\neq\begin{bmatrix}0\\2\end{bmatrix}=\begin{bmatrix}0\\2\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}=T\left(\begin{bmatrix}2\\0\end{bmatrix}\right)+T\left(\begin{bmatrix}0\\2\end{bmatrix}\right).$$

5 Let $p\in\operatorname{Ker}(D^2)$, so $D^2(p)=0$. Hence $D(D(p))=0$, which indicates that $D(p)$ must be a constant polynomial, and so $D(p)=a$ for some $a\in\mathbb{R}$. Now $p=\int a\,dx=ax+b$ for some arbitrary $b\in\mathbb{R}$, and thus $p\in\mathcal{P}_1(\mathbb{R})$. Conversely, if $p\in\mathcal{P}_1(\mathbb{R})$, so that $p=ax+b$ for some $a,b\in\mathbb{R}$, then

$$D^2(p)=D(D(p))=D(ax+b)=D(a)=0,$$

showing that $p\in\operatorname{Ker}(D^2)$. Therefore $\operatorname{Ker}(D^2)=\mathcal{P}_1(\mathbb{R})$.

In similar fashion we find that $\operatorname{Ker}(D^n)=\mathcal{P}_{n-1}(\mathbb{R})$, the subspace of all polynomials in x with real coefficients and degree at most $n-1$.

6 The only solution the system has is the zero vector in $\mathbb{R}^3$, $\mathbf{0} = [0,0,0]^\top$, and so the only basis for the space of solutions is $\varnothing$ (the empty set). Thus the dimension of the space of solutions is 0, by definition.

7 Let $\mathcal{E} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$ and $\mathcal{F} = (\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ denote the standard basis for $\mathbb{R}^4$ and $\mathbb{R}^3$, respectively. Since

$$L(\mathbf{e}_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad L(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad L(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad L(\mathbf{e}_4) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

the $\mathcal{EF}$-matrix for L is

$$[L]_{\mathcal{EF}} = \begin{bmatrix} [\mathbf{e}_1]_{\mathcal{E}} & [\mathbf{e}_2]_{\mathcal{E}} & [\mathbf{e}_3]_{\mathcal{E}} & [\mathbf{e}_4]_{\mathcal{E}} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

8a Let $\mathbf{v} \in V$. Then $P(P(\mathbf{v})) = (P \circ P)(\mathbf{v}) = P(\mathbf{v})$, showing that $P(\mathbf{v}) \in \mathrm{Img}(P)$. Now, $\mathbf{v} = (\mathbf{v} - P(\mathbf{v})) + P(\mathbf{v})$, where

$$P(\mathbf{v} - P(\mathbf{v})) = P(\mathbf{v}) - P(P(\mathbf{v})) = P(\mathbf{v}) - P(\mathbf{v}) = \mathbf{0}$$

shows that $\mathbf{v} - P(\mathbf{v}) \in \mathrm{Ker}(P)$. Thus $\mathbf{v} = (\mathbf{v} - P(\mathbf{v})) + P(\mathbf{v}) \in \mathrm{Ker}(P) + \mathrm{Img}(P)$, so that $V \subseteq \mathrm{Ker}(P) + \mathrm{Img}(P)$. The reverse containment is clear, and therefore $V = \mathrm{Ker}(P) + \mathrm{Img}(P)$.

8b Suppose $\mathbf{v} \in \mathrm{Ker}(P) \cap \mathrm{Img}(P)$, so $P(\mathbf{v}) = \mathbf{0}$ and there exists some $\mathbf{u} \in V$ such that $P(\mathbf{u}) = \mathbf{v}$. Now,

$$\mathbf{v} = P(\mathbf{u}) = P(P(\mathbf{u})) = P(\mathbf{v}) = \mathbf{0},$$

and thus $\mathbf{v} \in \{\mathbf{0}\}$. Since $P(\mathbf{0}) = \mathbf{0}$, it is clear that $\{\mathbf{0}\} \subseteq \mathrm{Ker}(P) \cap \mathrm{Img}(P)$, and therefore $\mathrm{Ker}(P) \cap \mathrm{Img}(P) = \{\mathbf{0}\}$.

9 Suppose $L(x,y) = [0,0]$. Then $2x + y = 0$ and $3x - 5y = 0$. These two equations form a system that has only one solution: $[x,y] = [0,0]$. Thus $\mathrm{Ker}(L) = \{[0,0]\}$, which implies that L is injective, and hence L is bijective (one-to-one and onto) since it is a linear operator on $\mathbb{R}^{1\times 2}$. A bijective function always has an inverse function, and therefore L is invertible.

10 Writing $I = -L - 2L = L \circ (-L - 2I)$ and $I = (-L - 2I) \circ L$, and so $-L - 2I$ is the inverse function for L. Therefore L is invertible.

Another way: suppose $\mathbf{v} \in \mathrm{Ker}(L)$, so that $L(\mathbf{v}) = \mathbf{0}$. Then, since $L(\mathbf{0}) = \mathbf{0}$, we have

$$\mathbf{0} = O(\mathbf{v}) = (L^2 + 2L + I)(\mathbf{v}) = L(L(\mathbf{v})) + 2L(\mathbf{v}) + I(\mathbf{v}) = L(\mathbf{0}) + 2\mathbf{0} + \mathbf{v} = \mathbf{v}.$$

Thus $\mathbf{v} = \mathbf{0}$, showing that $\mathrm{Ker}(L) = \{\mathbf{0}\}$, and therefore L is invertible.