MATH 260 EXAM #1 KEY (SUMMER 2016)

1a $\|\mathbf{u}\| = \sqrt{2^2 + (-1)^2 + 4^2} = \sqrt{21}$ and $\|\mathbf{v}\| = \sqrt{(-3)^2 + 1^2 + 2^2} = \sqrt{14}$.

1b Since $\mathbf{u} \cdot \mathbf{v} = (2)(-3) + (-1)(1) + (4)(2) = 1$ and $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = (\sqrt{14})^2 = 14$, we have

$$\text{proj}_{\mathbf{v}}\, \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v} = \tfrac{1}{14}[-3, 1, 2] = \left[-\tfrac{3}{14}, \tfrac{1}{14}, \tfrac{1}{7}\right].$$

1c We have

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{1}{\sqrt{21}\sqrt{14}} = \frac{1}{7\sqrt{6}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{1}{7\sqrt{6}}\right) \approx 86.66^\circ.$$

2 There's just one right answer:

$$\mathbf{x}(t) = \mathbf{p} + t(\mathbf{q} - \mathbf{p}) = [2, -6, 9] + t[-2, 14, -8] = [2 - 2t, -6 + 14t, 9 - 8t].$$

3 Let $\mathbf{p}_0 = [0, 1, 2]$, $\mathbf{p}_1 = [2, 3, 4]$, and $\mathbf{p}_2 = [4, 5, 6]$. We find a vector $\mathbf{n} = [a, b, c]$ such that $\mathbf{n} \perp (\mathbf{p}_1 - \mathbf{p}_0)$ and $\mathbf{n} \perp (\mathbf{p}_2 - \mathbf{p}_0)$; that is,

$$\mathbf{n} \cdot (\mathbf{p}_1 - \mathbf{p}_0) = 0 \quad \text{and} \quad \mathbf{n} \cdot (\mathbf{p}_2 - \mathbf{p}_0) = 0,$$

giving the system

$$\begin{cases} 2a + 2b + 2c = 0 \\ 4a + 4b + 4c = 0 \end{cases}$$

One solution is $\mathbf{n} = [a, b, c] = [1, -1, 0]$ (there are others). An equation for the plane is $(\mathbf{x} - \mathbf{p}_0) \cdot \mathbf{n} = 0$, or

$$[x, y - 1, z - 2] \cdot [1, -1, 0] = 0,$$

which becomes $x - y = -1$.

4 Two points determine the line, and they can be found as solutions to the system

$$\begin{cases} x - \ \ y + z = 3 \\ 2x - 3y - z = 1 \end{cases}$$

The first equation gives $z = 3 - x + y$, which when put into the second equation gives $y = \frac{3}{4}x - 1$. Putting this back into $z = 3 - x + y$ gives $z = -\frac{1}{4}x + 2$. The solution set of the system is

$$S = \left\{\left[x, \tfrac{3}{4}x - 1, -\tfrac{1}{4}x + 2\right] : x \in \mathbb{R}\right\}.$$

Replacing x with $4t$ simplifies things a little:

$$S = \left\{[4t, 3t - 1, -t + 2] : t \in \mathbb{R}\right\} = \left\{[0, -1, 2] + t[4, 3, -1] : t \in \mathbb{R}\right\}.$$

Thus

$$\mathbf{x}(t) = [0, -1, 2] + t[4, 3, -1]$$

is a parametric equation for the line of intersection of the two planes.

5 We have

$$\begin{bmatrix} -19 & 41 & 30 \\ -1 & 4 & 6 \\ 15 & -45 & -54 \end{bmatrix}$$

6a We wish to determine a, b, c, and d so that

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbf{I}.$$

If we let $a = 0$ and $d = 0$, then we find we must have b and c such that

$$\begin{bmatrix} bc & 0 \\ 0 & bc \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

which is to say $bc = -1$. Thus we may set $b = 1$ and $c = -1$ to obtain

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Another possibility is to let $b = c = 0$ and $a = d = i$, where $i = \sqrt{-1}$, giving

$$\mathbf{A} = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

There are infinitely many other options.

6b Find a, b, c, and d such that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & d^2+bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

or

$$a^2+bc=0, \quad ab+bd=0, \quad ac+cd=0, \quad d^2+bc=0.$$

The first and last equations require $a^2 = d^2 = -bc$, and so we must have $d = \pm a$. The second equation,

$$b(a+d) = 0,$$

requires $b = 0$ or $d = -a$. The third equation,

$$c(a+d) = 0,$$

requires $c = 0$ or $d = -a$.

Consider what happens if $d = -a$ for $a \neq 0$. Then $b, c \neq 0$, and we must have $c = -a^2/b$ to yield a matrix of the form

$$\begin{bmatrix} a & b \\ -a^2/b & -a \end{bmatrix}. \tag{1}$$

If $d = a$ for $a \neq 0$, then we must have $b = c = 0$ to satisfy the 2nd and 3rd equations; however, from the first and fourth equations we obtain $a^2 = d^2 = -bc = 0$, which requires $a = 0$ after all! This is an impossible case.

If $d = a = 0$, then $-bc = a^2 = 0$ results, and so we must have either $b = 0$ or $c = 0$, giving a matrix of the form

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}.$$

The first of these matrices may be obtained from (1) by setting $a = 0$ and so is not a new form. The second matrix is new. Therefore a 2×2 matrix $\mathbf{A}$ for which $\mathbf{A}^2 = \mathbf{O}$ must have one of these two forms:

$$\begin{bmatrix} a & b \\ -a^2/b & -a \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix}.$$

7 Call the matrix $\mathbf{A}$. Then,

$$\mathbf{A} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{bmatrix} \xrightarrow{-r_1+r_3 \to r_3} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{bmatrix} \xrightarrow{-2r_2+r_3 \to r_3} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

8 Performing row operations on

$$\left[\begin{array}{ccc|ccc} 0 & 3 & 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 & 0 \\ 2 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$$

until $\mathbf{I}_3$ is obtained on the left side, we find that

$$\mathbf{C}^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 \\ -\frac{2}{3} & 1 & 0 \end{bmatrix}$$

9 The corresponding augmented matrix for the system is

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 2 & 0 & -1 & -2 \\ 3 & 5 & 2 & 22 \end{array}\right].$$

We transform this matrix into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 2 & 0 & -1 & -2 \\ 3 & 5 & 2 & 22 \end{array}\right] \xrightarrow[-3r_1+r_3 \to r_3]{-2r_1+r_2 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -4 & 1 & -20 \\ 0 & -1 & 5 & -5 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -1 & 5 & -5 \\ 0 & -4 & 1 & -20 \end{array}\right]$$

$$\xrightarrow{-4r_2+r_3 \to r_3} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 9 \\ 0 & -1 & 5 & -5 \\ 0 & 0 & -19 & 0 \end{array}\right].$$

We have obtained the equivalent system of equations

$$\begin{cases} x + 2y - z = 9 \\ - y + 5z = -5 \\ - 19z = 0 \end{cases}$$

From the third equation we have $z = 0$, which when put into the second equation yields $-y = -5$, or $y = 5$. Finally, from the first equation we obtain.

$$x + 2(5) - 0 = 9 \quad \Rightarrow \quad x = -1.$$

Therefore the sole solution to the system is $(-1, 5, 0)$.

10 From 2nd equation: $x = 2z - y - 1$. Put this into 1st equation:

$$3(2z - y - 1) - 5y + 6z = 4 \quad \Rightarrow \quad y = \tfrac{3}{2}z - \tfrac{7}{8}.$$

Now $x = 2z - y - 1$ gives

$$x = 2z - \left(\tfrac{3}{2}z - \tfrac{7}{8}\right) - 1 = \tfrac{1}{2}z - \tfrac{1}{8}.$$

Solution set is therefore

$$\left\{ \begin{bmatrix} \frac{1}{2}z - \frac{1}{8} \\ \frac{3}{2}z - \frac{7}{8} \\ z \end{bmatrix} : z \in \mathbb{R} \right\}.$$

11 Note that $[1, 1, 1] \in S$, and yet $[-1, -1, -1] \notin S$. So not every element of S has an additive inverse, violating Axiom VS4.

12 Let $\mathbf{x}_1 = [x_1, y_1]$ and $\mathbf{x}_2 = [x_2, y_2]$ be elements of S, so $4y_1 - 3x_1 = 0$ and $4y_2 - 3x_2 = 0$. Now,

$$4(y_1 + y_2) - 3(x_1 + x_2) = (4y_1 - 3x_1) + (4y_2 - 3x_2) = 0,$$

which shows that $\mathbf{x}_1 + \mathbf{x}_2 = [x_1 + x_2, y_1 + y_2] \in S$. Also

$$4(cy_1) - 3(cx_1) = c(4y_1 - 3x_1) = (c)(0) = 0$$

shows that $c\mathbf{x}_1 \in S$. Since the closure properties are satisfied and $S \neq \varnothing$ (since $\mathbf{0} \in S$), we conclude that S is a subspace of $\mathbb{R}^2$.

13 Suppose $\mathbf{x}, \mathbf{y} \in U \cap V$ and c is a scalar. Since $\mathbf{x}, \mathbf{y} \in U$ and U is a subspace, we have $c\mathbf{x} \in U$ and $\mathbf{x} + \mathbf{y} \in U$. Since $\mathbf{x}, \mathbf{y} \in V$ and V is a subspace, we have $c\mathbf{x} \in V$ and $\mathbf{x} + \mathbf{y} \in V$. Therefore $c\mathbf{x} \in U \cap V$ and $\mathbf{x} + \mathbf{y} \in U \cap V$. We have now shown that $U \cap V$ is closed under scalar multiplication and vector addition, and therefore $U \cap V$ is a subspace.

14 Suppose that $\mathbf{x}, \mathbf{y} \in U^\perp$, so that $\mathbf{x} \cdot \mathbf{u} = 0$ and $\mathbf{y} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in U$. Since for all $\mathbf{u} \in U$ we have

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} + \mathbf{y} \cdot \mathbf{u} = 0 + 0 = 0,$$

it follows that $\mathbf{x} + \mathbf{y} \in U^\perp$.

Now suppose that $\mathbf{x} \in U^\perp$ and $c \in \mathbb{R}$. Then for any $\mathbf{u} \in U$ we have

$$(c\mathbf{x}) \cdot \mathbf{u} = c(\mathbf{x} \cdot \mathbf{u}) = c(0) = 0,$$

which implies that $c\mathbf{x} \in U^\perp$.

So, $U^\perp \neq \varnothing$ since $\mathbf{0} \in U^\perp$, and since $U^\perp$ is closed under scalar multiplication and vector addition it is a subspace of $\mathbb{R}^n$.

15 Let $\mathbf{x}_1 = [1, 2, 6]$, $\mathbf{x}_2 = [1, 5, -1]$, and $\mathbf{x}_3 = [0, 3, 1]$. Suppose $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 = \mathbf{0}$. This gives the system of equations

$$\begin{cases} c_1 + c_2 = 0 \\ 2c_1 + 5c_2 + 3c_3 = 0 \\ 6c_1 - c_2 + c_3 = 0 \end{cases}$$

The only solution to this system is $c_1 = c_2 = c_3 = 0$, and therefore $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a linearly independent set of vectors.