Math 260 Exam #3 Key (Summer 2015)

1 Expand along the second row to get

$$(-1)^{2+3}(-1)\begin{vmatrix} -1 & -3 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -3 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & -1 \end{vmatrix} = (-1)^{2+3}(-1)\begin{vmatrix} -1 & -3 \\ 1 & -1 \end{vmatrix} = 4.$$

2 Expanding along the first column, we have

$$\det(\mathbf{A}) = \begin{vmatrix} 0 & 4 \\ 3 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -12 - 2(-5) = -2.$$

Also, letting $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$,

$$\det(\mathbf{b}\ \mathbf{a}_2\ \mathbf{a}_3) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 0 & 4 \\ 3 & 3 & 1 \end{vmatrix} = 10, \quad \det(\mathbf{a}_1\ \mathbf{b}\ \mathbf{a}_3) = \begin{vmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \\ 0 & 3 & 1 \end{vmatrix} = 0,$$

and

$$\det(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 0 & 3 & 3 \end{vmatrix} = -6.$$

By Cramer's Rule the solution to the system is

$$(x_1, x_2, x_3) = \left(\frac{\det(\mathbf{b}\ \mathbf{a}_2\ \mathbf{a}_3)}{\det(\mathbf{A})}, \frac{\det(\mathbf{a}_1\ \mathbf{b}\ \mathbf{a}_3)}{\det(\mathbf{A})}, \frac{\det(\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{b})}{\det(\mathbf{A})}\right) = (-5, 0, 3).$$

3a Characteristic polynomial is

$$P_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}) = \begin{vmatrix} 3-t & 2 \\ 2 & 3-t \end{vmatrix} = (3-t)^2 - 4 = (t-5)(t-1),$$

and so the eigenvalues of $\mathbf{A}$ are 1, 5.

3b For the eigenvalue 1 the associated eigenspace is the solution set for $\mathbf{A}\mathbf{x} = \mathbf{x}$, where

$$\mathbf{A}\mathbf{x} = \mathbf{x} \ \Rightarrow\ (\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0} \ \Rightarrow\ \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields $x_2 = -x_1$. Hence

$$E_{\mathbf{A}}(1) = \left\{t\begin{bmatrix} 1 \\ -1 \end{bmatrix} : t \in \mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

For the eigenvalue 5 the associated eigenspace is the solution set for $\mathbf{A}\mathbf{x} = 5\mathbf{x}$, where

$$\mathbf{A}\mathbf{x} = 5\mathbf{x} \ \Rightarrow\ (\mathbf{A} - 5\mathbf{I})\mathbf{x} = \mathbf{0} \ \Rightarrow\ \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which yields $x_2 = x_1$. Hence

$$E_{\mathbf{A}}(5) = \left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

3c Let

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}.$$

It is routine to verify that

$$\mathbf{PDP}^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \mathbf{A}.$$

3d We have

$$\mathbf{A}^{50} = (\mathbf{PDP}^{-1})^{50} = \mathbf{PD}^{50}\mathbf{P}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 5^{50} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+5^{50} & -1+5^{50} \\ -1+5^{50} & 1+5^{50} \end{bmatrix}.$$

Next, let

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{5} \end{bmatrix},$$

and note that $(\mathbf{PCP}^{-1})^2 = \mathbf{PC}^2\mathbf{P}^{-1} = \mathbf{PDP}^{-1} = \mathbf{A}$. Hence

$$\mathbf{A}^{1/2} = \mathbf{PCP}^{-1} = \frac{1}{2}\begin{bmatrix} 1+\sqrt{5} & -1+\sqrt{5} \\ -1+\sqrt{5} & 1+\sqrt{5} \end{bmatrix}.$$

4a Applying the Gram-Schmidt Process,

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix},$$

and

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

An orthogonal basis for W is therefore

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\},$$

the latter basis obtained by replacing $\mathbf{w}_2$ with $2\mathbf{w}_2$ to rid ourselves of fractions.

4b Find the norms of the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ found above:

$$\|\mathbf{w}_1\| = \sqrt{2}, \quad \|\mathbf{w}_2\| = \frac{1}{\sqrt{2}}, \quad \|\mathbf{w}_3\| = \sqrt{10}.$$

An orthonormal basis for W is thus

$$\left\{ \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} \right\} = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{10}} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}.$$