Math 260 Exam #2 Key (Summer 2015)

1 Supposing $\mathbf{u}+\mathbf{w}=\mathbf{v}+\mathbf{w}$, we obtain $(\mathbf{u}+\mathbf{w})+(-\mathbf{w})=(\mathbf{v}+\mathbf{w})+(-\mathbf{w})$, so that $\mathbf{u}+(\mathbf{w}+(-\mathbf{w}))=\mathbf{v}+(\mathbf{w}+(-\mathbf{w}))$ by VS2, then $\mathbf{u}+\mathbf{0}=\mathbf{v}+\mathbf{0}$ by VS4, and finally $\mathbf{u}=\mathbf{v}$ by VS3.

2 Consider $\mathbf{u}=[1,0,0]^\top$ and $\mathbf{v}=[0,1,0]^\top$. We have $\|\mathbf{u}\|=1=1+0=|u_1|+|u_2|$ and $\|\mathbf{v}\|=1=0+1=|v_1|+|v_2|$, so $\mathbf{u},\mathbf{v}\in U$. But $\mathbf{w}=\mathbf{u}+\mathbf{v}=[1,1,0]^\top$, so that

$$\|\mathbf{w}\|=\sqrt{1^2+1^2+0^2}=\sqrt{2}\neq 2=|w_1|+|w_2|,$$

which shows that $\mathbf{u}+\mathbf{v}=\mathbf{w}\notin U$. Hence U is not closed under vector addition and cannot be a subspace.

3 We must find $x_1,x_2,x_3\in\mathbb{R}$ such that $x_1\mathbf{u}_1+x_2\mathbf{u}_2+x_3\mathbf{u}_3=\mathbf{v}$, if possible. This gives us the system

$$\begin{cases} 4x_1+4x_2+x_3=7\\ 2x_1-3x_2+3x_3=16\\ x_1+2x_2-x_3=-3\end{cases}$$

This system has the unique solution $(x_1,x_2,x_3)=(2,-1,3)$.

4a We can do elementary column operations on $\mathbf{A}$ to get the column rank (and hence the rank) of the matrix:

$$\mathbf{A}\xrightarrow[-c_1+c_3\to c_3]{-2c_1+c_2\to c_2}\begin{bmatrix}1&0&0\\1&1&2\\0&2&4\end{bmatrix}\xrightarrow{-2c_2+c_3\to c_3}\begin{bmatrix}1&0&0\\1&1&0\\0&2&0\end{bmatrix}.$$

We now see that $\text{rank}(\mathbf{A})=2$, and since column operations do not alter a column space, we also have a basis for $\text{Col}(\mathbf{A})$:

$$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}0\\1\\2\end{bmatrix}\right\}.$$

But it is quick to verify that the first two columns of $\mathbf{A}$ are linearly independent, and since we now know that $\text{rank}(\mathbf{A})=2$ it follows that they also constitute a basis for $\text{Col}(\mathbf{A})$:

$$\left\{\begin{bmatrix}1\\1\\0\end{bmatrix},\begin{bmatrix}2\\3\\2\end{bmatrix}\right\}.$$

Other bases are possible.

4b By definition $\text{Nul}(\mathbf{A})=\{\mathbf{x}\in\mathbb{R}^3:\mathbf{A}\mathbf{x}=\mathbf{0}\}$, otherwise known as the solution set of the system

$$\begin{cases} x_1+2x_2+x_3=0\\ x_1+3x_2+3x_3=0\\ 2x_2+4x_3=0\end{cases}$$

This system is equivalent to

$$\begin{cases} x_1 + 2x_2 + \;\; x_3 = 0 \\ \qquad\;\; x_2 + 2x_3 = 0 \\ \qquad\;\; x_2 + 2x_3 = 0 \end{cases}$$

From this we obtain $x_2 = -2x_3$ and $x_1 = 3x_3$, and hence

$$\text{Nul}(\mathbf{A}) = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} : x_1 = 3x_3, \;\; x_2 = -2x_3 \right\} = \left\{ t \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

This shows that $[3, -2, 1]^\top$ is a basis for $\text{Nul}(\mathbf{A})$. Any nonzero scalar multiple of this vector will also work.

5 Suppose that $\mathbf{x} \in \text{Nul}(\mathbf{B})$, so that $\mathbf{Bx} = \mathbf{0}$. Recalling that matrix multiplication is associative, we find that

$$(\mathbf{AB})\mathbf{x} = \mathbf{A}(\mathbf{Bx}) = \mathbf{A0} = \mathbf{0},$$

which shows $\mathbf{x} \in \text{Nul}(\mathbf{AB})$. Hence $\text{Nul}(\mathbf{B}) \subseteq \text{Nul}(\mathbf{AB})$.

Now suppose that $\mathbf{x} \in \text{Nul}(\mathbf{AB})$, so that $(\mathbf{AB})\mathbf{x} = \mathbf{0}$. Now, since $\mathbf{A}$ is invertible, we find that

$$(\mathbf{AB})\mathbf{x} = \mathbf{0} \;\;\Rightarrow\;\; \mathbf{A}^{-1}(\mathbf{AB})\mathbf{x} = \mathbf{A}^{-1}\mathbf{0} \;\;\Rightarrow\;\; (\mathbf{A}^{-1}\mathbf{A})\mathbf{Bx} = \mathbf{0} \;\;\Rightarrow\;\; \mathbf{IBx} = \mathbf{0} \;\;\Rightarrow\;\; \mathbf{Bx} = \mathbf{0},$$

which shows $\mathbf{x} \in \text{Nul}(\mathbf{B})$. Hence $\text{Nul}(\mathbf{AB}) \subseteq \text{Nul}(\mathbf{B})$, and therefore $\text{Nul}(\mathbf{AB}) = \text{Nul}(\mathbf{B})$.

6 The top row of $\mathbf{C}$ has nonzero first component while the bottom row has first component equal to zero, which means the two row vectors are linearly independent and the dimension of $\text{Row}(\mathbf{C})$ is 2. Thus the dimension of $\text{Col}(\mathbf{C})$ must also be 2. Since $\text{rank}(\mathbf{C}) = 2$ and the Rank-Nullity Theorem informs us that $\text{rank}(\mathbf{C}) + \text{nullity}(\mathbf{C}) = 5$, we find that $\text{nullity}(\mathbf{C}) = 3$. That is, the dimension of $\text{Nul}(\mathbf{C})$ is 3.

7a Letting $\mathbf{e}_1 = [1, 0]^\top$ and $\mathbf{e}_2 = [0, 1]^\top$, by an established theorem we have

$$\mathbf{I}_{\mathcal{EB}} = \big[[\mathbf{e}_1]_{\mathcal{B}} \;\;\; [\mathbf{e}_2]_{\mathcal{B}} \big],$$

and so we must find the $\mathcal{B}$-coordinates of $\mathbf{e}_1$ and $\mathbf{e}_2$. Letting $\mathbf{b}_1 = [1, 2]^\top$ and $\mathbf{b}_2 = [-2, 1]^\top$, so that $\mathcal{B} = (\mathbf{b}_1, \mathbf{b}_2)$, we must find $x_1, x_2, y_1, y_2 \in \mathbb{R}$ so $x_1\mathbf{b}_1 + x_2\mathbf{b}_2 = \mathbf{e}_1$ and $y_1\mathbf{b}_1 + y_2\mathbf{b}_2 = \mathbf{e}_2$; that is,

$$\begin{cases} \;\, x_1 - 2x_2 = 1 \\ 2x_1 + \;\; x_2 = 0 \end{cases} \qquad \text{and} \qquad \begin{cases} \;\, y_1 - 2y_2 = 0 \\ 2y_1 + \;\; y_2 = 1 \end{cases}$$

Solving these systems gives $(x_1, x_2) = (\frac{1}{5}, -\frac{2}{5})$ and $(y_1, y_2) = (\frac{2}{5}, \frac{1}{5})$. Thus $[\mathbf{e}_1]_{\mathcal{B}} = [\frac{1}{5}, -\frac{2}{5}]^\top$ and $[\mathbf{e}_2]_{\mathcal{B}} = [\frac{2}{5}, \frac{1}{5}]^\top$, and we obtain

$$\mathbf{I}_{\mathcal{EB}} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}.$$

7b We have

$$[\mathbf{x}]_{\mathcal{B}} = \mathbf{I}_{\mathcal{E}\mathcal{B}}[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 2 \\ -5 \end{bmatrix} = \begin{bmatrix} -\frac{8}{5} \\ -\frac{9}{5} \end{bmatrix}.$$

8a Let $[L]$ denote the matrix corresponding to L with respect to the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$. Setting

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

we have

$$[L]\mathbf{B} = \left[[L]\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad [L]\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}. \tag{1}$$

The square matrix $\mathbf{B}$ has linearly independent columns and so is invertible. Right-multiplying through (1) by $\mathbf{B}^{-1}$ gives

$$[L] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \mathbf{B}^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

8b We have

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = [L]\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \\ y \end{bmatrix}.$$

8c The range (i.e. image) of L is

$$\mathrm{Img}(L) = \left\{ \begin{bmatrix} s \\ t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\},$$

which is $\mathrm{Col}([L])$ as to be expected, and happens to be the plane $y = z$ in $\mathbb{R}^3$.

The kernel (i.e. null space) of L is

$$\mathrm{Nul}(L) = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : L\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : \begin{bmatrix} x \\ y \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Thus the kernel of L is trivial, which straightaway implies that L is one-to-one.