Math 260 Exam #1 Key (Summer 2015)

1 We have

$$\cos\theta = \frac{\mathbf{p}\cdot\mathbf{q}}{\|\mathbf{p}\|\|\mathbf{q}\|} = \frac{-20}{\sqrt{26}\sqrt{38}} = -\frac{10}{\sqrt{247}} \quad\Rightarrow\quad \theta = \arccos\left(-\frac{10}{\sqrt{247}}\right) \approx 129.5^\circ.$$

2 Set $x_0 = (4,5,1)$ and $x_1 = (1,3,-2)$, and let $\mathbf{v} = \overrightarrow{x_1x_0}$. Then

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathbf{v} \quad\Rightarrow\quad \mathbf{x}(t) = \begin{bmatrix}4\\5\\1\end{bmatrix} + t\begin{bmatrix}3\\2\\3\end{bmatrix}$$

for $t \in \mathbb{R}$.

3 A normal vector for the plane is $\mathbf{n} = [2,1,-3]^\top$ (which is a direction vector for the given line). Since the plane contains the point $o = (0,0,0)$, the plane will consist of all points (x,y,z) for which the vector $[x,y,z]^\top - [0,0,0]^\top = [x,y,z]^\top$ is orthogonal to $\mathbf{n} = [2,1,-3]^\top$:

$$[2,1,-3]^\top \cdot [x,y,z]^\top = 0 \quad\Rightarrow\quad 2x + y - 3z = 0.$$

This is the nonparametric equation for the plane. The nonparametric equation can be used to find two other points on the plane such as $p_0 = (1,-2,0)$ and $p_1 = (0,3,1)$. Let

$$\mathbf{u} = \mathbf{p}_0 - \mathbf{0} = \begin{bmatrix}1\\-2\\0\end{bmatrix} \quad\text{and}\quad \mathbf{v} = \mathbf{p}_1 - \mathbf{0} = \begin{bmatrix}0\\3\\1\end{bmatrix}.$$

A parametric equation is

$$\mathbf{p}(t) = \mathbf{0} + s\mathbf{u} + t\mathbf{v} \quad\Rightarrow\quad \mathbf{p}(t) = s\begin{bmatrix}1\\-2\\0\end{bmatrix} + t\begin{bmatrix}0\\3\\1\end{bmatrix}$$

for $s,t \in \mathbb{R}$.

4 Letting $x = y = 0$ in the plane's equation gives $z = 1/7$, so $p = (0,0,1/7)$ is a point on the plane. Let $\mathbf{v} = \overrightarrow{pq} = \mathbf{q} - \mathbf{p} = [5,2,-\frac{22}{7}]^\top$. A normal vector for the plane is $\mathbf{n} = [4,-4,7]^\top$. We project $\mathbf{v}$ onto $\mathbf{n}$:

$$\operatorname{proj}_{\mathbf{n}}(\mathbf{v}) = \left(\frac{\mathbf{v}\cdot\mathbf{n}}{\mathbf{n}\cdot\mathbf{n}}\right)\mathbf{n} = -\tfrac{10}{81}\begin{bmatrix}4\\-4\\7\end{bmatrix}.$$

The distance D is the magnitude of this vector:

$$D = \|\operatorname{proj}_{\mathbf{n}}(\mathbf{v})\| = \tfrac{10}{9}.$$

5 The corresponding augmented matrix for the system is

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & -2 \end{array}\right].$$

We transform this matrix into row-echelon form:

$$\left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ -2 & 3 & -1 & 0 \\ -6 & 6 & 0 & -2 \end{array}\right] \xrightarrow[6r_1+r_3\to r_3]{2r_1+r_2\to r_2} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 3 & -3 & 2 \\ 0 & 6 & -6 & 4 \end{array}\right] \xrightarrow{-2r_2+r_3\to r_3} \left[\begin{array}{rrr|r} 1 & 0 & -1 & 1 \\ 0 & 3 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We have obtained the equivalent system of equations

$$\begin{cases} x - \ z = \ 1 \\ 3y - 3z = 2, \end{cases}$$

giving $x = z + 1$ and $y = z + \frac{2}{3}$. Any ordered triple (x, y, z) that satisfies the original system must be of the form

$$(z + 1, 3z + \tfrac{2}{3}, z)$$

for some $z \in \mathbb{R}$, and therefore the solution set is

$$\left\{(z + 1, 3z + \tfrac{2}{3}, z) : z \in \mathbb{R}\right\}.$$

In terms of column vectors, we have

$$\begin{bmatrix} z+1 \\ z+\frac{2}{3} \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \\ 0 \end{bmatrix} + \begin{bmatrix} z \\ z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \\ 0 \end{bmatrix} + z\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and so solution set is

$$\left\{ \frac{1}{3}\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\},$$

which is a line in $\mathbb{R}^3$.

6 We employ the same sequence of elementary row operations on both $\mathbf{A}$ and $\mathbf{I}_3$, as follows.

$$\left[\begin{array}{rrrr|r} 3 & -6 & -1 & 1 & 7 \\ -1 & 2 & 2 & 3 & 1 \\ 4 & -8 & -3 & -2 & 6 \end{array}\right] \xrightarrow{r_2\leftrightarrow r_1} \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 1 \\ 3 & -6 & -1 & 1 & 7 \\ 4 & -8 & -3 & -2 & 6 \end{array}\right] \xrightarrow[4r_1+r_3\to r_3]{3r_1+r_2\to r_2}$$

$$\left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 5 & 10 & 10 \\ 0 & 0 & 5 & 10 & 10 \end{array}\right] \xrightarrow[r_1/5\to r_1]{-r_2+r_3\to r_3} \left[\begin{array}{rrrr|r} -1 & 2 & 2 & 3 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We now have the equivalent system of equations

$$\begin{cases} -x + 2y + 2z + 3w = 1 \\ z + 2w = 2 \end{cases}$$

giving $z = 2 - 2w$ and $x = 2y - w + 3$ for $y, w \in \mathbb{R}$. Letting $s = y$ and $t = w$, solution set is

$$\left\{ \begin{bmatrix} 2s - t + 3 \\ s \\ 2 - 2t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\},$$

which is a plane in $\mathbb{R}^4$.

7 Letting $\mathbf{b} = [b_1, b_2, b_3, b_4]^\top$, we have

$$\left[\begin{array}{rrr|l} 1 & 0 & -1 & b_1 \\ -2 & 3 & -1 & b_2 \\ 3 & -3 & 0 & b_3 \\ 2 & 0 & -2 & b_4 \end{array}\right] \xrightarrow[\substack{-3r_1+r_3\to r_3 \\ -2r_1+r_4\to r_4}]{2r_1+r_2\to r_2} \left[\begin{array}{rrr|l} 1 & 0 & -1 & b_1 \\ 0 & 3 & -3 & b_2 + 2b_1 \\ 0 & -3 & 3 & b_3 - 3b_1 \\ 0 & 0 & 0 & b_4 - 2b_1 \end{array}\right] \xrightarrow{r_2+r_3\to r_3} \left[\begin{array}{rrr|l} 1 & 0 & -1 & b_1 \\ 0 & 3 & -3 & b_2 + 2b_1 \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \\ 0 & 0 & 0 & b_4 - 2b_1 \end{array}\right],$$

so we must have $b_4 - 2b_1 = 0$ and $b_3 + b_2 - b_1 = 0$, which implies $b_4 = 2b_1$ and $b_3 = b_1 - b_2$. We have

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 - b_2 \\ 2b_1 \end{bmatrix}.$$

8 The product $\mathbf{AB}$ is undefined, while

$$\mathbf{BA} = \begin{bmatrix} 3 & -6 & 9 & -12 \\ 2 & -4 & 6 & -8 \\ 1 & -2 & 3 & -4 \end{bmatrix}.$$

9 One possibility:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

10 Letting

$$\mathbf{X} = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \end{bmatrix},$$

the equation $\mathbf{XA} = \mathbf{I}$ (where $\mathbf{I}$ is the 2×2 identity matrix) yields a system of four equations in six unknowns:

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 = 1 \\ -x_1 - x_2 + 2x_3 = 0 \\ 2x_4 + 4x_5 + 2x_6 = 0 \\ -x_4 - x_5 + 2x_6 = 1 \end{cases}$$

This system breaks into two separate systems:

$$\begin{cases} 2x_1 + 4x_2 + 2x_3 = 1 \\ -x_1 - x_2 + 2x_3 = 0 \end{cases} \qquad \begin{cases} 2x_4 + 4x_5 + 2x_6 = 0 \\ -x_4 - x_5 + 2x_6 = 1 \end{cases}$$

With some algebra we find that

$$x_2 = -\tfrac{3}{5}x_1 + \tfrac{1}{5}, \quad x_3 = \tfrac{1}{5}x_1 + \tfrac{1}{10}, \quad x_5 = -\tfrac{3}{5}x_4 - \tfrac{1}{5}, \quad x_6 = \tfrac{1}{5}x_4 + \tfrac{2}{5},$$

and so

$$\mathbf{X} = \begin{bmatrix} x_1 & -\frac{3}{5}x_1 + \frac{1}{5} & \frac{1}{5}x_1 + \frac{1}{10} \\ x_4 & -\frac{3}{5}x_4 - \frac{1}{5} & \frac{1}{5}x_4 + \frac{2}{5} \end{bmatrix}.$$

If desired one may write

$$\mathbf{X}(s,t) = \begin{bmatrix} s & -\frac{3}{5}s + \frac{1}{5} & \frac{1}{5}s + \frac{1}{10} \\ t & -\frac{3}{5}t - \frac{1}{5} & \frac{1}{5}t + \frac{2}{5} \end{bmatrix}$$

for $s, t \in \mathbb{R}$.