Math 260 Exam #3 Key (Summer 2014)

1a To find the $\mathcal{B}$-coordinates of $\mathbf{v}$, we find $a, b \in \mathbb{R}$ such that

$$a\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \\ 3 \end{bmatrix},$$

which is to say we solve the system

$$\begin{cases} -a + \ \ b = 5 \\ \ \ a + 2b = 7 \\ \ \ a + \ \ b = 3 \end{cases}$$

The only solution is $(a, b) = (-1, 4)$, and therefore

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

1b Letting $\mathbf{u}_1 = [-1\ 1\ 1]^\top$ and $\mathbf{u}_2 = [1\ 2\ 1]^\top$, by an established theorem we have

$$\mathbf{M} = \Big[[\mathbf{u}_1]_{\mathcal{C}} \ \ [\mathbf{u}_2]_{\mathcal{C}} \Big],$$

and so we must find the $\mathcal{C}$-coordinates of $\mathbf{u}_1$ and $\mathbf{u}_2$. Starting with $\mathbf{u}_1$, we find $a, b \in \mathbb{R}$ such that $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{u}_1$; that is,

$$a\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

which has $(a, b) = (1, 1)$ as the only solution, and hence

$$[\mathbf{u}_1]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Next, we find $a, b \in \mathbb{R}$ such that $a\mathbf{v}_1 + b\mathbf{v}_2 = \mathbf{u}_2$; that is,

$$a\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},$$

which has $(a, b) = (2, 1)$ as the only solution, and hence

$$[\mathbf{u}_2]_{\mathcal{C}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{M} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

1c Using the $\mathcal{B}$-coordinates of $\mathbf{v}$ found above, we have

$$[\mathbf{v}]_{\mathcal{C}} = \mathbf{M}[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}.$$

2 We have $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$, and $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. Now,

$$L(\mathbf{v}_1) = L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -5(3) + 13(1) \\ -7(3) + 16(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}.$$

We need the $\mathcal{C}$-coordinates of $L(\mathbf{v}_1)$, which means finding a_1, a_2, a_3 such that

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = L(\mathbf{v}_1);$$

that is,

$$\begin{cases} a_1 - a_2 = 1 \\ 2a_2 + a_3 = -2 \\ -a_1 + 2a_2 + 2a_3 = -5, \end{cases}$$

which solves to give $a_1 = 1$, $a_2 = 0$, and $a_3 = -2$. Thus

$$[L(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

Next,

$$L(\mathbf{v}_2) = L\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

We need the $\mathcal{C}$-coordinates of $L(\mathbf{v}_2)$, so we find a_1, a_2, a_3 such that

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = L(\mathbf{v}_2).$$

Like before, this yields a system of equations. We put its augmented matrix into row-echelon form:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ -1 & 2 & 2 & -3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

which solves to give $a_1 = 3$, $a_2 = 1$, and $a_3 = -1$. Thus

$$[L(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

The $\mathcal{BC}$-matrix of L is therefore

$$[L]_{\mathcal{BC}} = \Big[\big[L(\mathbf{v}_1)\big]_{\mathcal{C}} \;\; \big[L(\mathbf{v}_2)\big]_{\mathcal{C}} \Big] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

3a Find the characteristic polynomial:

$$P_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}) = \begin{vmatrix} 2-t & 0 & -2 \\ 0 & 3-t & 0 \\ 0 & 0 & 3-t \end{vmatrix} = (2-t)\begin{vmatrix} 3-t & 0 \\ 0 & 3-t \end{vmatrix} = (2-t)(3-t)^2.$$

The characteristic equation is $(2-t)(3-t)^2 = 0$, which has solution set $\{2, 3\}$. Hence the eigenvalues of $\mathbf{A}$ are 2 and 3.

3b The eigenspace corresponding to 2 is

$$E_{\mathbf{A}}(2) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{A}\mathbf{x} = 2\mathbf{x}\} = \{\mathbf{x} : (\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0}\}.$$

Passing to the augmented matrix for the system $(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0}$, we have

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Thus the solution set of the system is

$$\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : y = z = 0 \text{ and } x \in \mathbb{R}\right\} = \left\{\begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}.$$

A basis for $E_{\mathbf{A}}(2)$ is thus $\mathcal{B}_1 = \{[1,0,0]^\top\}$.

The eigenspace corresponding to 3 is

$$E_{\mathbf{A}}(3) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{A}\mathbf{x} = 3\mathbf{x}\} = \{\mathbf{x} : (\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}\}.$$

Passing to the augmented matrix for the system $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$, we have

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the solution set of the system is

$$\left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = -2z \text{ and } y, z \in \mathbb{R}\right\} = \left\{s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R}\right\} = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right\}.$$

A basis for $E_{\mathbf{A}}(3)$ is thus $\mathcal{B}_2 = \{[0,1,0]^\top, [-2,0,1]^\top\}$.

3c A spectral basis for $\mathbf{A}$ (i.e. a basis for $\mathbb{R}^3$ consisting of linearly independent eigenvectors of $\mathbf{A}$) is the ordered basis

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\right).$$

The eigenvalues corresponding to these eigenvalues are 2, 3, and 3, respectively. Therefore the diagonal matrix we seek is

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

As for $\mathbf{P}$, that is the 3×3 matrix with column vectors being the vectors in $\mathcal{B}$ in the order that they appear:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

4a By Ye Olde Gram-Schmidt Process,

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix},$$

and

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

An orthogonal basis for W is therefore

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\} \quad \text{or} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\},$$

the latter basis obtained by replacing $\mathbf{w}_2$ with $2\mathbf{w}_2$ to rid ourselves of fractions.

4b Find the norms of the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ found above:

$$\|\mathbf{w}_1\| = \sqrt{2}, \quad \|\mathbf{w}_2\| = \frac{1}{\sqrt{2}}, \quad \|\mathbf{w}_3\| = \sqrt{10}.$$

An orthonormal basis for W is thus

$$\left\{ \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} \right\} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}.$$