Math 260 Exam #2 Key (Summer 2014)

1a Put augmented matrix for system in row-echelon form:

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 3 \\ 1 & -1 & 2 & 3 \\ 1 & -2 & \lambda & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & \lambda & 4 \\ 1 & -1 & 2 & 3 \\ 2 & 1 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & \lambda & 4 \\ 0 & 1 & 2-\lambda & -1 \\ 0 & 5 & 1-2\lambda & -5 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & \lambda & 4 \\ 0 & 1 & 2-\lambda & -1 \\ 0 & 0 & 3\lambda-9 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 4-\lambda & 2 \\ 0 & 1 & 2-\lambda & -1 \\ 0 & 0 & 3-\lambda & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 3-\lambda & 0 \end{array}\right].$$

From this it can be seen that there is no value for λ that results in a system having no solution.

1b If $\lambda \neq 3$ there will be a unique solution. The third equation in the system obtained above is $(3-\lambda)z = 0$, which gives $z = 0$ when $\lambda \neq 3$. The second equation $y - z = -1$ then yields $y = -1$, and the first equation $x + z = 2$ yields $x = 2$. Thus

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

is the unique solution for *any* $\lambda \neq 3$.

1c If $\lambda = 3$ there will be infinitely many solutions. The third equation in the system above becomes $0z = 0$ when $\lambda = 3$, and hence z can be any real number. The second equation gives $y = z - 1$, and the first equation gives $x = 2 - z$. Solution set to the system is therefore

$$\left\{ \begin{bmatrix} 2-z \\ z-1 \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$

2a Certainly W_1 contains $\mathbf{O}_2$, the 2×2 matrix with all entries 0, and hence $W_1 \neq \varnothing$. Suppose that $\mathbf{A}, \mathbf{B} \in W_1$, so that $\mathbf{A}^\top = \mathbf{A}$ and $\mathbf{B}^\top = \mathbf{B}$. Since

$$(\mathbf{A} + \mathbf{B})^\top = \mathbf{A}^\top + \mathbf{B}^\top = \mathbf{A} + \mathbf{B},$$

we have $\mathbf{A} + \mathbf{B} \in W_1$. Also, for any $c \in \mathbb{R}$,

$$(c\mathbf{A})^\top = c\mathbf{A}^\top = c\mathbf{A},$$

and so $c\mathbf{A} \in W_1$. Since W_1 is a nonempty set that is closed under vector addition and scalar multiplication, we conclude that it is a subspace of $\mathrm{Mat}_2(\mathbb{R})$.

2b Observe that

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in W_2,$$

since we need only choose the real numbers $a = 1$ and $b = 0$ to obtain

$$\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

On the other hand

$$-4\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 0 & 0 \end{bmatrix} \notin W_2,$$

since there is no real number a for which $a^2 = -4$! Therefore W_2 is not a subspace of $\mathrm{Mat}_2(\mathbb{R})$, since it is not closed under scalar multiplication.

3 The set $S = \{\mathbf{u}_1, \mathbf{u}_2\}$ does not span $\mathbb{R}^2$ since, for instance, $[1 \;\; 0]^\top$ is not in $\mathrm{Span}(S)$. From

$$a\mathbf{u}_1 + b\mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

comes the system

$$\begin{cases} -a + 2b = 1 \\ 3a - 6b = 0 \end{cases}$$

which is readily found to have no solution (multiply the first equation by -3 to see that the system is inconsistent).

4a We can show the vectors are linearly independent by showing they form a basis for $\mathbb{R}^3$. Define the matrix $\mathbf{A} = [\mathbf{u}_1 \;\; \mathbf{u}_2 \;\; \mathbf{u}_3]$, which is the matrix with $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ as its column vectors. In the textbook there is a theorem that implies that the column vectors of $\mathbf{A}$ are a basis for $\mathbb{R}^3$ if and only if $\det(\mathbf{A}) \neq 0$. We have

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 & -2 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 0 & 3 & 2 \\ 0 & 1 & 3 \\ -1 & 0 & 2 \end{vmatrix} = -7 \neq 0,$$

and so $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent.

4b Find $a, b, c \in \mathbb{R}$ such that $a\mathbf{u}_1 + b\mathbf{u}_2 + \mathbf{u}_3 = \mathbf{v}$. This gives the system

$$\begin{cases} 2a + 3b - 2c = -6 \\ \quad\quad\; b + 3c = -10 \\ -a \quad\quad + 2c = -5 \end{cases}$$

Solving the system gives $(a, b, c) = (1, -4, -2)$, so

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -4 \\ -2 \end{bmatrix}.$$

5a The yz-plane, P_{yz}, is the set of points in $\mathbb{R}^3$ that satisfy the equation $x = 0$. That is,

$$P_{yz} = \left\{ \begin{bmatrix} 0 \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\} = \left\{ s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

From this we see that

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

is a basis for P_{yz}.

5b We have $x = 3z - 2y$, so the plane P consists of points (x, y, z) such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3z - 2y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix},$$

where $y, z \in \mathbb{R}$. That is,

$$P = \operatorname{Span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

The vectors

$$\mathbf{v}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

are readily verified to be linearly independent, and therefore $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for P.