Math 260 Exam #1 Key (Summer 2014)

1a We have

$$\begin{bmatrix} 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = [14],$$

which in practice is identified with the scalar 14.

1b We have

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 9 & -3 & 6 \\ -3 & 1 & -2 \\ 6 & -2 & 4 \end{bmatrix}$$

1c We have

$$\mathbf{AC} = \begin{bmatrix} -2 & -9 \\ -12 & 3 \\ 9 & 7 \end{bmatrix}$$

2 First we find the inverse of the 2×2 matrix on the righthand side of the equation, using the equation

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

to get

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{(1)(3) - (-2)(-1)} \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}.$$

Now,

$$5\mathbf{A} - \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = 3\mathbf{A} + \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} \Rightarrow 2\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 1 & 6 \end{bmatrix} \Rightarrow \mathbf{A} = \begin{bmatrix} 2 & 2 \\ 1/2 & 3 \end{bmatrix}.$$

3 If we let

$$\mathbf{p} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} -3 \\ -2 \\ -3 \end{bmatrix},$$

then one vector equation for the line is $\mathbf{x} = \mathbf{p} + t\mathbf{v}$ for $t \in \mathbb{R}$, or

$$\mathbf{x} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} - t \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Other representations are possible.

4a It can help to get the vector equation (also known as the parameterization) of L_2. Letting $z = 1$ gives $x = 5$ and

$$y = \frac{z-1}{2} + 4 = \frac{1-1}{2} + 4 = 4,$$

so $(5, 4, 1)$ is one point on L_2. Letting $z = 3$ gives $x = 5$ and

$$y = \frac{z-1}{2} + 4 = \frac{3-1}{2} + 4 = 5,$$

so $(5, 5, 3)$ is another point on L_2. Letting

$$\mathbf{p} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}, \quad \mathbf{q} = \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \mathbf{q} - \mathbf{p} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

then L_2 is given by

$$\mathbf{x}_2(t) = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \quad t \in \mathbb{R},$$

while L_1 is given by

$$\mathbf{x}_1(s) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \quad s \in \mathbb{R}.$$

We must find $s, t \in \mathbb{R}$ such that $\mathbf{x}_1(s) = \mathbf{x}_2(t)$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

This gives rise to the system of equations

$$\begin{cases} 2s + 1 = 5 \\ s + 1 = t + 4 \\ -s + 1 = 2t + 1 \end{cases}$$

which has the unique solution $(s, t) = (2, -1)$. That is,

$$\mathbf{x}_2(-1) = \mathbf{x}_1(2) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix},$$

so $(5, 3, -1)$ is the point of intersection.

4b The plane certainly contains all points that L_1 and L_2 contain, such as

$$\mathbf{p}_1 = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{p}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{p}_3 = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}.$$

Letting

$$\mathbf{u} = \mathbf{p}_2 - \mathbf{p}_1 = \begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \mathbf{p}_3 - \mathbf{p}_1 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix},$$

a vector equation (or parameterization) for the plane is $\mathbf{x} = \mathbf{p}_1 + s\mathbf{u} + t\mathbf{v}$, $(s,t) \in \mathbb{R}^2$. That is,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} + s\begin{bmatrix} -4 \\ -2 \\ 2 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}. \tag{1}$$

This is not the "equation" for the plane. To find the equation, we obtain from (1) the system

$$\begin{cases} x = 5 - 4s \\ y = 3 - 2s + t \\ z = -1 + 2s + 2t \end{cases}$$

From the system's first and second equations we have

$$s = \frac{5-x}{4} \quad \text{and} \quad t = y - 3 + 2s = y - 3 + \frac{5-x}{4},$$

Substituting these into the third equation then yields

$$z = -1 + 2s + 2t = -1 + 2\left(\frac{5-x}{4}\right) + 2\left(y - 3 + \frac{5-x}{4}\right) = \frac{1}{2} + 2y - \frac{3}{2}x,$$

or

$$3x - 4y + 2z = 1,$$

which is the equation of the plane.

5a Let $p = (5,1,3)$. Geometrically, the plane P may be characterized as the set of all points $q = (x,y,z) \in \mathbb{R}^3$ such that the vector $\overrightarrow{pq}$ is orthogonal to $\mathbf{n}$, which is to say $\overrightarrow{pq} \cdot \mathbf{n} = 0$. Now,

$$0 = \overrightarrow{pq} \cdot \mathbf{n} = \begin{bmatrix} x-5 & y-1 & z-3 \end{bmatrix}^\top \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} = (x-5) - 4(y-1) + 2(z-3),$$

and thus

$$x - 4y + 2z = 7$$

is the algebraic equation of P.

5b From the algebraic equation we have $x = 7 - 4y + 2z$, and so P may be characterized as the set of vectors of the form

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 - 4y + 2z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix};$$

that is, P is given as the vector equation

$$\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

6 The corresponding augmented matrix for the system is

$$\left[\begin{array}{rrr|r} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{array}\right].$$

We transform this matrix into row-echelon form:

$$\left[\begin{array}{rrr|r} -3 & -5 & 36 & 10 \\ -1 & 0 & 7 & 5 \\ 1 & 1 & -10 & -4 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_3} \left[\begin{array}{rrr|r} 1 & 1 & -10 & -4 \\ -1 & 0 & 7 & 5 \\ -3 & -5 & 36 & 10 \end{array}\right] \xrightarrow[3r_1+r_3 \to r_3]{r_1+r_2 \to r_2} \left[\begin{array}{rrr|r} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & -2 & 6 & -2 \end{array}\right]$$

$$\xrightarrow{2r_2+r_3 \to r_3} \left[\begin{array}{rrr|r} 1 & 1 & -10 & -4 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

We have obtained the equivalent system of equations

$$\begin{cases} x + y - 10z = -4 \\ \quad\;\; y - \;\; 3z = \;\; 1 \end{cases}$$

From the second equation we have

$$y = 3z + 1,$$

which, when substituted into the first equation, yields

$$x = 10z - y - 4 = 10z - (3z + 1) - 4 = 7z - 5.$$

That is, we have $x = 7z - 5$ and $y = 3z + 1$, and z is free to assume any scalar value whatsoever.

Any ordered triple (x, y, z) that satisfies the original system must be of the form

$$(7z - 5, 3z + 1, z)$$

for some $z \in \mathbb{R}$, and therefore the solution set is

$$S = \{(7z - 5, 3z + 1, z) : z \in \mathbb{R}\}.$$

In terms of column vectors, we have

$$\begin{bmatrix} 7z - 5 \\ 3z + 1 \\ z \end{bmatrix} = \begin{bmatrix} 7z \\ 3z \\ z \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} = z \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix},$$

and so

$$S = \left\{ \begin{bmatrix} -5 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \;:\; t \in \mathbb{R} \right\}.$$

7 We employ the same sequence of elementary row operations on both $\mathbf{A}$ and $\mathbf{I}_3$, as follows.

$$\left[\begin{array}{rrr|rrr} 2 & 4 & 3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-r_1 \to r_1]{r_2 \leftrightarrow r_1} \left[\begin{array}{lrr|lrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2r_1+r_2 \to r_2}$$

$$\left[\begin{array}{lrr|lrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 10 & 3 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{lrr|lrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 10 & 3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{-5r_2+r_3 \to r_3}$$

$$\left[\begin{array}{lrr|lrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow{\frac{1}{2}r_2 \to r_2} \left[\begin{array}{lrr|lrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow{3r_2+r_1 \to r_1}$$

$$\left[\begin{array}{llr|lrr} 1 & 0 & 3/2 & 0 & -1 & 3/2 \\ 0 & 1 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow[\frac{3}{4}r_3+r_1 \to r_1]{\frac{1}{4}r_3+r_2 \to r_2} \left[\begin{array}{llr|rrr} 1 & 0 & 0 & 3/4 & 1/2 & -9/4 \\ 0 & 1 & 0 & 1/4 & 1/2 & -3/4 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow{-\frac{1}{2}r_3 \to r_3}$$

$$\left[\begin{array}{lll|rrr} 1 & 0 & 0 & 3/4 & 1/2 & -9/4 \\ 0 & 1 & 0 & 1/4 & 1/2 & -3/4 \\ 0 & 0 & 1 & -1/2 & -1 & 5/2 \end{array}\right].$$

Therefore

$$\mathbf{A}^{-1} = \begin{bmatrix} 3/4 & 1/2 & -9/4 \\ 1/4 & 1/2 & -3/4 \\ -1/2 & -1 & 5/2 \end{bmatrix}.$$

To solve $\mathbf{Ax} = \mathbf{b}$ we use $\mathbf{A}^{-1}$:

$$\mathbf{Ax} = \mathbf{b} \quad \Rightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{4}\begin{bmatrix} 3 & 2 & -9 \\ 1 & 2 & -3 \\ -2 & -4 & 10 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 15 \\ 5 \\ -14 \end{bmatrix} = \begin{bmatrix} 15/4 \\ 5/4 \\ -7/2 \end{bmatrix}.$$

8 Let $\mathbf{C}$ denote the matrix. Then

$$\det(\mathbf{C}) \xlongequal[-2r_1+r_4 \to r_4]{\substack{-2r_1+r_2 \to r_2 \\ -r_1+r_3 \to r_3}} \begin{vmatrix} 1 & -4 & 3 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 6 & 3 & -2 \\ 0 & -2 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 \\ 6 & 3 & -2 \\ -2 & 8 & 0 \end{vmatrix}$$

$$\xlongequal{4c_1+c_2 \to c_2} \begin{vmatrix} 1 & 3 & -3 \\ 6 & 27 & -2 \\ -2 & 0 & 0 \end{vmatrix} = (-1)^{3+1}(-2)\begin{vmatrix} 3 & -3 \\ 27 & -2 \end{vmatrix}$$

$$= -2[(3)(-2) - (-3)(27)] = -150.$$

9 Here $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ -1 & 2 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix}.$$

We have

$$\det(\mathbf{A}) = \begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix} - \begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 3 - 1 + 5 = 7,$$

and by Cramer's Rule,

$$x = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} 8 & 1 & 1 \\ 3 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix} = \frac{1}{7}(21) = 3$$

$$y = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} 1 & 8 & 1 \\ 2 & 3 & -1 \\ -1 & 3 & 1 \end{bmatrix} = \frac{1}{7}(7) = 1$$

$$z = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} 1 & 1 & 8 \\ 2 & 1 & 3 \\ -1 & 2 & 3 \end{bmatrix} = \frac{1}{7}(28) = 4.$$

The solution is therefore $(x, y, z) = (3, 1, 4)$.

10 $\mathbf{A}$ is not invertible if and only if $\det(\mathbf{A}) = 0$; that is, we need

$$\begin{vmatrix} 2-\lambda & 3 \\ 2 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) - (3)(2) = (\lambda - 4)(\lambda + 1) = 0,$$

which leads us to conclude that $\mathbf{A}$ is not invertible if and only if $\lambda = -1, 4$.