Math 260 Exam #4 Key (Summer 2013)

1 Here expansion according to the first column will be done:

$$\begin{vmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 3 & 1 & 5 & 7 \end{vmatrix} \xrightarrow{3r_1+r_3\to r_3} \begin{vmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 4 & 11 & 7 \end{vmatrix} = (-1)^{1+1}(-1)\begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 2 \\ 4 & 11 & 7 \end{vmatrix} \xrightarrow[-7r_1+r_3\to r_3]{-2r_1+r_2\to r_2}$$

$$-\begin{vmatrix} 3 & 2 & 1 \\ -2 & -3 & 0 \\ -17 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ -2 & -3 & 0 \\ 17 & 3 & 0 \end{vmatrix} = (-1)^{1+3}(1)\begin{vmatrix} -2 & -3 \\ 17 & 3 \end{vmatrix} = (-2)(3) - (-3)(17)$$

$$= -6 + 51 = 45.$$

2 Every 3×3 subdeterminant of the given matrix $\mathbf{A}$ equals 0, so no three of the four column vectors of $\mathbf{A}$ are linearly independent and therefore $\operatorname{rank}(\mathbf{A}) \le 2$. However, the submatrix

$$\mathbf{B} = \begin{bmatrix} 3 & 5 & 1 & 4 \\ 2 & -1 & 1 & 1 \end{bmatrix}$$

has nonzero subdeterminant

$$\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = 1 - 4 = -3,$$

which is to say $\mathbf{B}$ has two linearly independent column vectors and so the two row vectors of $\mathbf{B}$ must also be linearly independent. Of course, the two row vectors of $\mathbf{B}$ are also row vectors of $\mathbf{A}$, implying that $\operatorname{rank}(\mathbf{A}) \ge 2$. Therefore $\operatorname{rank}(\mathbf{A}) = 2$.

3 Here $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We have

$$\det(\mathbf{A}) = 2\begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} = 2(-3) - 2 + 4(-1) = -12,$$

and by Cramer's Rule,

$$x = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -\frac{1}{12}(-5) = \frac{5}{12}$$

$$y = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{vmatrix} = -\frac{1}{12}(1) = -\frac{1}{12}$$

$$z = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 0 \\ 4 & -3 & 2 \end{vmatrix} = -\frac{1}{12}(-1) = \frac{1}{12}$$

4 We set out to find all

$$\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

for which

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

for some $\lambda \in \mathbb{R}$. The matrix equation corresponds to the system of equations

$$\begin{cases} (1-\lambda)x + \qquad ay = 0 \\ \qquad\qquad (1-\lambda)y = 0 \end{cases}$$

Now, if $\lambda \neq 1$ then we must have $x = y = 0$, but $\mathbf{0}$ by definition cannot be an eigenvector. Assume that $\lambda = 1$. Then the system consists only of the equation $ay = 0$, and since $a \neq 0$ we must have $y = 0$ while x remains arbitrary. Thus the set of eigenvectors for the matrix

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

is

$$E = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\},$$

which is easily seen to be a subspace of $\mathbb{R}^2$ with basis

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}.$$

Since $|\mathcal{B}| = 1$ we conclude that E is a 1-dimensional vector space.

5

6 We have

$$P_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}_3) = \det\left(\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} - \begin{bmatrix} t & 0 & 0 \\ 0 & t & 0 \\ 0 & 0 & t \end{bmatrix} \right) = \begin{vmatrix} 1-t & -3 & 3 \\ 3 & -5-t & 3 \\ 6 & -6 & 4-t \end{vmatrix}$$

$$\xrightarrow{c_1+c_2 \to c_2} \begin{vmatrix} 1-t & -2-t & 3 \\ 3 & -2-t & 3 \\ 6 & 0 & 4-t \end{vmatrix} \xrightarrow{-r_1+r_2 \to r_2} \begin{vmatrix} 1-t & -2-t & 3 \\ t+2 & 0 & 0 \\ 6 & 0 & 4-t \end{vmatrix}.$$

Expanding the determinant according to the 2nd row then gives

$$P_{\mathbf{A}}(t) = (-1)^{2+1}(t+2) \begin{vmatrix} -2-t & 3 \\ 0 & 4-t \end{vmatrix} = (t+2)^2(t-4),$$

and so we see that $P_{\mathbf{A}}(t) = 0$ for $t = -2, 4$. Thus the eigenvalues of $\mathbf{A}$ are $\lambda = -2, 4$.

The eigenspace of $\mathbf{A}$ corresponding to $\lambda = -2$ is

$$E_{-2} = \mathrm{Nul}(\mathbf{A} + 2\mathbf{I}_3) = \{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{A} + 2\mathbf{I}_3)\mathbf{x} = \mathbf{0}\}$$

$$= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Writing the matrix equation—which is a homogeneous system of equations—as an augmented matrix, we have

$$\left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right] \xrightarrow[-2r_1+r_3\to r_3]{-r_1+r_2\to r_2} \left[\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}r_1\to r_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Hence $x_1 - x_2 + x_3 = 0$, which implies that $x_3 = x_2 - x_1$ and so

$$E_{-2} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0 \right\} = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_2 - x_1 \end{bmatrix} : x_1, x_2 \in \mathbb{R} \right\}.$$

Observing that

$$\begin{bmatrix} x_1 \\ x_2 \\ x_2 - x_1 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ -x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} x_2,$$

we have

$$E_{-2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} x_2 : x_1, x_2 \in \mathbb{R} \right\}$$

and so it is clear that

$$\mathcal{B}_{-2} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is a linearly independent set of vectors that spans E_{-2} and therefore must be a basis for E_{-2}. Notice that the elements of $\mathcal{B}_{-2}$ are in fact eigenvectors of $\mathbf{A}$ corresponding to the eigenvalue -2, as are all the vectors belonging to E_{-2}.

Next, the eigenspace of $\mathbf{A}$ corresponding to $\lambda = 4$ is

$$\begin{aligned} E_4 &= \mathrm{Nul}(\mathbf{A} - 4\mathbf{I}_3) = \{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{A} - 4\mathbf{I}_3)\mathbf{x} = \mathbf{0}\} \\ &= \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 : \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}. \end{aligned}$$

Applying Gaussian Elimination to the corresponding augmented matrix yields

$$\left[\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right] \xrightarrow[2r_1+r_3\to r_3]{r_1+r_2\to r_2} \left[\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & -12 & 6 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}r_2+r_1\to r_1]{-r_2+r_3\to r_3} \left[\begin{array}{ccc|c} -3 & 3 & 0 & 0 \\ 0 & -12 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

From the top row we obtain $x_2 = x_1$, and from the middle row we obtain $x_3 = 2x_2$ and thus $x_3 = 2x_1$. Now,

$$E_4 = \left\{ \begin{bmatrix} x_1 \\ x_1 \\ 2x_1 \end{bmatrix} : x_1 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} x_1 : x_1 \in \mathbb{R} \right\}.$$

Clearly

$$\mathcal{B}_4 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

is a linearly independent set that spans E_4 and so qualifies as a basis for E_4. The vector belonging to $\mathcal{B}_4$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue 4, as is any real scalar multiple of the vector.