Math 260 Exam #2 Key (Summer 2013)

1 This is easy: the set does not contain the zero vector ($0^2 + 0^2 + 0^2 \neq 1$), which violates Axiom A3.

2 The set is a vector space—and to prove it is not hard, just tedious (there are eight axioms, after all). Let

$$S = \{f : \mathbb{R} \to \mathbb{R} \mid f(10) = 0\}.$$

Let $f, g, h \in S$ and $a, b \in \mathbb{R}$. Let $x \in \mathbb{R}$ be arbitrary. We have $f + g \in S$ and $af \in S$ since

$$(f + g)(10) = f(10) + g(10) = 0 + 0 = 0 \quad \text{and} \quad (af)(10) = af(10) = a(0) = 0,$$

and so there is closure under addition and scalar multiplication.

By the Commutative Property of Addition we have

$$(f + g)(x) = f(x) + g(x) = g(x) + f(x) = (g + f)(x),$$

so that $f + g = g + f$. Axiom A1 holds.

We have

$$f(x) + [g(x) + h(x)] = [f(x) + g(x)] + h(x)$$

by the Associative Property of Addition, and thus $f + (g + h) = (f + g) + h$. Axiom A2 holds.

Let o be the zero function. That is, $o(x) = 0$ for all $x \in \mathbb{R}$. Since $o(10) = 0$ we see that $o \in S$. Now,

$$(o + f)(x) = o(x) + f(x) = 0 + f(x) = f(x)$$

and

$$(f + o)(x) = f(x) + o(x) = f(x) + 0 = f(x),$$

and so $o + f = f + o = f$. Axiom A3 holds.

As usual $-f$ is the function given by $(-f)(x) = -f(x)$, so in particular $(-f)(10) = -f(10) = 0$ implies that $-f \in S$. Now,

$$(-f + f)(x) = (-f)(x) + f(x) = -f(x) + f(x) = 0 = o(x)$$

shows that $-f + f = o$. Similarly $f + (-f) = o$. Axiom A4 holds.

By the Distributive Property,

$$\begin{aligned}\big(a(f + g)\big)(x) &= a(f + g)(x) = a[f(x) + g(x)] = af(x) + ag(x)\\ &= (af)(x) + (ag)(x) = (af + ag)(x),\end{aligned}$$

which shows that $a(f + g) = af + ag$. Axiom A5 holds.

Again by the Distributive Property,

$$\big((a + b)f\big)(x) = (a + b)f(x) = af(x) + bf(x) = (af)(x) + (bf)(x) = (af + bf)(x),$$

so $(a + b)f = af + bf$. Axiom A6 holds.

By the Associative Property of Multiplication,

$$\big(a(bf)\big)(x) = a(bf)(x) = a(bf(x)) = (ab)f(x) = \big((ab)f\big)(x),$$

so $a(bf) = (ab)f$. Axiom A7 holds.

Finally, since $1 \in \mathbb{R}$ is the multiplicative identity, we have $(1f)(x) = 1f(x) = f(x)$. This shows that $1f = f$, and Axiom A8 is jumping for joy.

3 Let S denote the set, and let $\langle x_1, y_1\rangle$ and $\langle x_2, y_2\rangle$ be elements of S. Let $c \in \mathbb{R}$. Since

$$cx_1 - 2(cy_1) = c(x_1 - 2y_1) = c(0) = 0,$$

we conclude that $\langle cx_1, cy_1\rangle \in S$. Also,

$$(x_1 + x_2) - 2(y_1 + y_2) = (x_1 - 2y_1) + (x_2 - 2y_2) = 0 + 0 = 0$$

shows that $\langle x_1 + x_2, y_1 + y_2\rangle \in S$. Since S is closed under scalar multiplication and vector addition (i.e. $\langle x_1, y_1\rangle \in S$ implies $c\langle x_1, y_1\rangle = \langle cx_1, cy_1\rangle \in S$, and $\langle x_1, y_1\rangle, \langle x_2, y_2\rangle \in S$ implies $\langle x_1, y_1\rangle + \langle x_2, y_2\rangle = \langle x_1 + x_2, y_1 + y_2\rangle \in S$), we conclude that S is a subspace of $\mathbb{R}^2$.

4 Suppose $\mathbf{x}, \mathbf{y} \in U \cap V$ and c is a scalar. Since $\mathbf{x}, \mathbf{y} \in U$ and U is a subspace, we have $c\mathbf{x} \in U$ and $\mathbf{x} + \mathbf{y} \in U$. Since $\mathbf{x}, \mathbf{y} \in V$ and V is a subspace, we have $c\mathbf{x} \in V$ and $\mathbf{x} + \mathbf{y} \in V$. Therefore $c\mathbf{x} \in U \cap V$ and $\mathbf{x} + \mathbf{y} \in U \cap V$. We have now shown that $U \cap V$ is closed under scalar multiplication and vector addition, and therefore $U \cap V$ is a subspace.

5 Suppose that $\mathbf{x}, \mathbf{y} \in U^\perp$, so that $\mathbf{x} \cdot \mathbf{u} = 0$ and $\mathbf{y} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in U$. Since for all $\mathbf{u} \in U$ we have

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} + \mathbf{y} \cdot \mathbf{u} = 0 + 0 = 0,$$

it follows that $\mathbf{x} + \mathbf{y} \in U^\perp$.

Now suppose that $\mathbf{x} \in U^\perp$ and $c \in \mathbb{R}$. Then for any $\mathbf{u} \in U$ we have

$$(c\mathbf{x}) \cdot \mathbf{u} = c(\mathbf{x} \cdot \mathbf{u}) = c(0) = 0,$$

which implies that $c\mathbf{x} \in U^\perp$.

So, $U^\perp \neq \varnothing$ since $\mathbf{0} \in U^\perp$, and since $U^\perp$ is closed under scalar multiplication and vector addition it is a subspace of $\mathbb{R}^n$.

6 We show that $S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} \geq c\}$ is convex. For any $\mathbf{u}, \mathbf{v} \in S$ let $\mathbf{x} \in \ell_{\mathbf{uv}}$, the line segment joining $\mathbf{u}$ and $\mathbf{v}$. Then $\mathbf{x} = (1 - t)\mathbf{u} + t\mathbf{v}$ for some $t \in [0, 1]$. Now, since $\mathbf{u}$ and $\mathbf{v}$ are in S we have $\mathbf{a} \cdot \mathbf{u} \geq c$ and $\mathbf{a} \cdot \mathbf{v} \geq c$, so that

$$\mathbf{a} \cdot \mathbf{x} = (1 - t)\mathbf{a} \cdot \mathbf{u} + t\mathbf{a} \cdot \mathbf{v} \geq (1 - t)c + tc = c$$

and therefore $\mathbf{x} \in S$. This shows that $\ell_{\mathbf{uv}} \subseteq S$, and since $\mathbf{u}, \mathbf{v} \in S$ are arbitrary it follows that S is convex.

7 The proposition in §3.4 of the Notes (also §3.4 of the book) could be used here: $\langle a, b\rangle$ and $\langle c, d\rangle$ are linearly independent if and only if $ad - bc \neq 0$. So, since

$$(1)(5) - (2)(1) = 3 \neq 0,$$

we conclude that $\langle 1, 2\rangle$ and $\langle 1, 5\rangle$ are linearly independent. (There are at least a couple other ways to do this problem.)

8 Let $c_1, c_2, c_3 \in \mathbb{R}$ be such that

$$c_1 \cos 2x + c_2 \cos^2 x + c_3 \sin^2 x = 0 \tag{1}$$

for all $x \in (-\infty, \infty)$. The functions $\cos 2x$, $\cos^2 x$, and $\sin^2 x$ are linearly independent on $(-\infty, \infty)$ if and only if the only way to satisfy (1) for all $-\infty < x < \infty$ is to have $c_1 = c_2 = c_3 = 0$. However, it is true that $\cos 2x = \cos^2 x - \sin^2 x$. Thus (1) becomes

$$c_1(\cos^2 x - \sin^2 x) + c_2 \cos^2 x + c_3 \sin^2 x = 0.$$

Now notice that this equation, and subsequently (1), is satisfied for all $-\infty < x < \infty$ if we let $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$. So (1) has a nontrivial solution on $(-\infty, \infty)$, and therefore the functions $\cos 2x$, $\cos^2 x$, and $\sin^2 x$ are linearly dependent on $(-\infty, \infty)$.

9 We get a matrix in row-echelon form using elementary row operations on the given matrix:

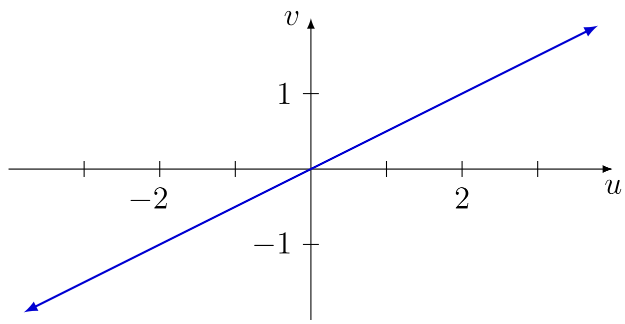
$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 2 & 1 \\ 3 & 4 & 2 & 3 \end{bmatrix} \xrightarrow[-3r_1+r_3 \to r_3]{-r_1+r_2 \to r_2} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{-r_2+r_3 \to r_3} \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in row-echelon form. We see that $\text{rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A}) = 2$.

10 T takes $(x, y) \in \mathbb{R}^2$ and returns $(u, v) \in \mathbb{R}^2$ given by $(u, v) = (xy, y)$. Points on the line $x = 2$ are of the form $(2, y)$, where $-\infty < y < \infty$, and so for these points T returns $(u, v) = (2y, y)$. We see that $y = v$, so in the uv-coordinate system the image under T of the line $x = 2$ is the set $\{(2v, v) : v \in \mathbb{R}\}$, or equivalently

$$\left\{ \left(v, \tfrac{1}{2}v\right) : v \in \mathbb{R} \right\}.$$

This is also a line, as pictured below.



11 Not linear. In general $T(cx, cy) \neq cT(x, y)$. Take the case when $c = 2$, $x = 1$, and $y = 2$, for instance: $T(2, 4) = (2)(4) = 8$, but $2T(1, 2) = 2(1)(2) = 4$. Hence $T(2, 4) \neq 2T(1, 2)$.

12 Let C be a convex set in a vector space V, and $L : V \to W$ a linear transformation. We must show that

$$L(C) = \{L(\mathbf{v}) : \mathbf{v} \in V\}$$

is a convex set.

Let $\mathbf{a}, \mathbf{b} \in L(C)$ be arbitrary and fix $\mathbf{x} \in \ell_{\mathbf{ab}}$ (the line segment joining $\mathbf{a}$ and $\mathbf{b}$), so that there exists some $t \in [0, 1]$ such that $\mathbf{x} = (1 - t)\mathbf{a} + t\mathbf{b}$. Now, there exist some $\boldsymbol{\alpha}, \boldsymbol{\beta} \in C$ such that $L(\boldsymbol{\alpha}) = \mathbf{a}$ and $L(\boldsymbol{\beta}) = \mathbf{b}$, and since C is convex the vector $\boldsymbol{\xi} = (1-t)\boldsymbol{\alpha} + t\boldsymbol{\beta}$ must be an element of C. Observing that

$$L(\boldsymbol{\xi}) = L((1-t)\boldsymbol{\alpha} + t\boldsymbol{\beta}) = (1-t)L(\boldsymbol{\alpha}) + tL(\boldsymbol{\beta}) = (1-t)\mathbf{a} + t\mathbf{b} = \mathbf{x}$$

for $\boldsymbol{\xi} \in C$, we conclude that $\mathbf{x} \in L(C)$. Hence $\ell_{\mathbf{ab}} \subseteq L(C)$ and $L(C)$ is convex.

13 Suppose S is a line in V, which is to say $S = \{\mathbf{u} + t\mathbf{v} : t \in \mathbb{R}\}$ for some $\mathbf{u}, \mathbf{v} \in V$. For $L : V \to W$ there are two cases: (1) $L(\mathbf{v}) = \mathbf{0}$; and (2) $L(\mathbf{v}) \neq \mathbf{0}$. Since $L(\mathbf{u} + t\mathbf{v}) = L(\mathbf{u}) + tL(\mathbf{v})$, we find that Case (1) results in a point in W: for any $t \in \mathbb{R}$ we get

$$L(\mathbf{u} + t\mathbf{v}) = L(\mathbf{u}) + tL(\mathbf{v}) = L(\mathbf{u}) + t\mathbf{0} = L(\mathbf{u}) + \mathbf{0} = L(\mathbf{u}),$$

and so $L(S) = \{L(\mathbf{u})\}$; that is, the image of S under L consists of the single point $L(\mathbf{u})$. On the other hand Case (2) results in the line

$$\{L(\mathbf{u}) + tL(\mathbf{v}) : t \in \mathbb{R}\}$$

in W.