Math 260 Exam #3 Key (Summer 2012)

1a. Let $\mathbf{v} \in V$. Now, $P(P(\mathbf{v})) = (P \circ P)(\mathbf{v}) = P(\mathbf{v})$, which implies that

$$P(\mathbf{v} - P(\mathbf{v})) = P(\mathbf{v}) - P(P(\mathbf{v})) = \mathbf{0}$$

and so $\mathbf{v} - P(\mathbf{v}) \in \text{Ker}(P)$. Noting that $P(\mathbf{v}) \in \text{Im}(P)$, we readily obtain

$$\mathbf{v} = (\mathbf{v} - P(\mathbf{v})) + P(\mathbf{v}) \in \text{Ker}(P) + \text{Im}(P).$$

Thus $V \subseteq \text{Ker}(P) + \text{Im}(P)$, and since the reverse containment follows from the closure properties of a vector space, we conclude that $V = \text{Ker}(P) + \text{Im}(P)$.

1b. Let $\mathbf{v} \in \text{Ker}(P) \cap \text{Im}(P)$. Then $P(\mathbf{v}) = \mathbf{0}$ and there exists some $\mathbf{u} \in V$ such that $P(\mathbf{u}) = \mathbf{v}$. With these results and the hypothesis $P \circ P = P$, we have

$$\mathbf{0} = P(\mathbf{v}) = P(P(\mathbf{u})) = P(\mathbf{u}) = \mathbf{v},$$

implying $\mathbf{v} \in \{\mathbf{0}\}$ and so $\text{Ker}(P) \cap \text{Im}(P) \subseteq \{\mathbf{0}\}$. The reverse containment holds since $\text{Ker}(P)$ and $\text{Im}(P)$ are subspaces of V and so must both contain $\mathbf{0}$. Therefore $\text{Ker}(P) \cap \text{Im}(P) = \{\mathbf{0}\}$.

2. Let $\mathbf{v} \in U + W$, so $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u} \in U$ and $\mathbf{w} \in W$. Now, let $\mathbf{u}' \in U$ and $\mathbf{w}' \in W$ be such that $\mathbf{v} = \mathbf{u}' + \mathbf{w}'$. Then

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (\mathbf{u} + \mathbf{w}) - (\mathbf{u}' + \mathbf{w}') = (\mathbf{u} - \mathbf{u}') + (\mathbf{w} - \mathbf{w}'),$$

and so $\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w}$. Since $\mathbf{u} - \mathbf{u}' \in U$ and $\mathbf{w}' - \mathbf{w} \in W$, it follows that $\mathbf{u} - \mathbf{u}', \mathbf{w}' - \mathbf{w} \in U \cap W$. But $U \cap W = \{\mathbf{0}\}$, so $\mathbf{u} - \mathbf{u}' = \mathbf{w}' - \mathbf{w} = \mathbf{0}$ and we conclude that $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$. That is, the vectors $\mathbf{u} \in U$ and $\mathbf{w} \in W$ for which $\mathbf{v} = \mathbf{u} + \mathbf{w}$ are unique.

3. Suppose that $L(x, y) = (0, 0)$. Then $2x + y = 0$ and $3x - 5y = 0$, which is a homogeneous system of equations which has only the trivial solution $x = y = 0$. That is, if $(x, y) \in \text{Ker}(L)$, then $(x, y) = (0, 0)$ and it follows that $\text{Ker}(L) = \{(0, 0)\}$. Therefore L is invertible.

4. Suppose that $\mathbf{v} \in V$ is such that $L(\mathbf{v}) = \mathbf{0}$. Then

$$\begin{aligned}\mathbf{0} = O(\mathbf{v}) &= (L^2 + 2L + I)(\mathbf{v}) = L^2(\mathbf{v}) + 2L(\mathbf{v}) + I(\mathbf{v}) \\ &= L(L(\mathbf{v})) + 2L(\mathbf{v}) + \mathbf{v} = L(\mathbf{0}) + 2\mathbf{0} + \mathbf{0} = \mathbf{0} + 2\mathbf{0} + \mathbf{v} = \mathbf{v},\end{aligned}$$

which implies that $\text{Ker}(L) = \{\mathbf{0}\}$ and so L is invertible.

5. Here expansion according to the first column will be done:

$$\begin{vmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 3 & 1 & 5 & 7 \end{vmatrix} \xrightarrow{3r_1 + r_3 \to r_3} \begin{vmatrix} -1 & 1 & 2 & 0 \\ 0 & 3 & 2 & 1 \\ 0 & 4 & 1 & 2 \\ 0 & 4 & 11 & 7 \end{vmatrix} = (-1)^{1+1}(-1) \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 2 \\ 4 & 11 & 7 \end{vmatrix} \xrightarrow[-7r_1 + r_3 \to r_3]{-2r_1 + r_2 \to r_2}$$

$$-\begin{vmatrix} 3 & 2 & 1 \\ -2 & -3 & 0 \\ -17 & -3 & 0 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ -2 & -3 & 0 \\ 17 & 3 & 0 \end{vmatrix} = (-1)^{1+3}(1)\begin{vmatrix} -2 & -3 \\ 17 & 3 \end{vmatrix} = (-2)(3) - (-3)(17)$$

$$= -6 + 51 = 45.$$

6. Every 3×3 subdeterminant of the given matrix $\mathbf{A}$ equals 0, so no three of the four column vectors of $\mathbf{A}$ are linearly independent and therefore $\text{rank}(\mathbf{A}) \leq 2$. However, the submatrix

$$\mathbf{B} = \begin{bmatrix} 3 & 5 & 1 & 4 \\ 2 & -1 & 1 & 1 \end{bmatrix}$$

has nonzero subdeterminant

$$\begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = 1 - 4 = -3,$$

which is to say $\mathbf{B}$ has two linearly independent column vectors and so the two row vectors of $\mathbf{B}$ must also be linearly independent. Of course, the two row vectors of $\mathbf{B}$ are also row vectors of $\mathbf{A}$, implying that $\text{rank}(\mathbf{A}) \geq 2$. Therefore $\text{rank}(\mathbf{A}) = 2$.

7. Here $\mathbf{Ax} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

We have

$$\det(\mathbf{A}) = 2\begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} = 2(-3) - 2 + 4(-1) = -12,$$

and by Cramer's Rule,

$$x = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -\frac{1}{12}(-5) = \frac{5}{12}$$

$$y = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{vmatrix} = -\frac{1}{12}(1) = -\frac{1}{12}$$

$$z = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 0 \\ 4 & -3 & 2 \end{vmatrix} = -\frac{1}{12}(-1) = \frac{1}{12}$$

8. We set out to find all $\langle x, y \rangle \in \mathbb{R}^2$ for which

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

for some $\lambda \in \mathbb{R}$. The matrix equation corresponds to the system of equations

$$\begin{cases} (1-\lambda)x + & ay = 0 \\ & (1-\lambda)y = 0 \end{cases}$$

Now, if $\lambda \neq 1$ then we must have $x = y = 0$, but $\langle 0, 0\rangle$ by definition cannot be an eigenvector. Assume that $\lambda = 1$. Then the system consists only of the equation $ay = 0$, and since $a \neq 0$ we must have $y = 0$ while x remains arbitrary. Thus the set of eigenvectors for the matrix

$$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

is $S = \{\langle x, 0\rangle : x \in \mathbb{R}\}$, which is easily seen to be a subspace of $\mathbb{R}^2$ with basis $\mathcal{B} = \{\langle 1, 0\rangle\}$. Since $|\mathcal{B}| = 1$ we conclude that S is a 1-dimensional vector space.

9a. We have

$$P_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}_3) = \begin{vmatrix} 4-t & 0 & 1 \\ -2 & 1-t & 0 \\ -2 & 0 & 1-t \end{vmatrix} = (1-t)\begin{vmatrix} 4-t & 1 \\ -2 & 1-t \end{vmatrix}$$

$$= (1-t)[(4-t)(1-t)+2] = (1-t)(t^2 - 5t + 6) = (1-t)(t-3)(t-2).$$

9b. We see that $P_{\mathbf{A}}(t) = 0$ for $t = 1, 2, 3$, so the eigenvalues of $\mathbf{A}$ are $\lambda = 1, 2, 3$.

9c. The eigenspace of $\mathbf{A}$ corresponding to $\lambda = 1$ is

$$E_1 = \text{Nul}(\mathbf{A} - \mathbf{I}_3) = \{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{A} - \mathbf{I}_3)\mathbf{x} = \mathbf{0}\},$$

which is the set of all $\mathbf{x} = \langle x_1, x_2, x_3\rangle$ such that

$$\begin{bmatrix} 3 & 0 & 1 \\ -2 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and so $E_1 = \{\langle 0, x_2, 0\rangle : x_2 \in \mathbb{R}\}$. A basis for E_1 is $\{\langle 0, 1, 0\rangle\}$.

The eigenspace of $\mathbf{A}$ corresponding to $\lambda = 2$ is

$$E_2 = \text{Nul}(\mathbf{A} - 2\mathbf{I}_3) = \{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{A} - 2\mathbf{I}_3)\mathbf{x} = \mathbf{0}\},$$

which is the set of all $\mathbf{x} = \langle x_1, x_2, x_3\rangle$ such that

$$\begin{bmatrix} 2 & 0 & 1 \\ -2 & -1 & 0 \\ -2 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so that we must have $x_2 = x_3$ and $x_1 = -\frac{1}{2}x_3$. Hence

$$E_2 = \left\{\left\langle -\tfrac{1}{2}x_3, x_3, x_3 \right\rangle : x_3 \in \mathbb{R}\right\}$$

A basis for E_2 is $\{\langle -1, 2, 2\rangle\}$.

The eigenspace of $\mathbf{A}$ corresponding to $\lambda = 3$ is

$$E_3 = \text{Nul}(\mathbf{A} - 3\mathbf{I}_3) = \{\mathbf{x} \in \mathbb{R}^3 : (\mathbf{A} - 3\mathbf{I}_3)\mathbf{x} = \mathbf{0}\},$$

which is the set of all $\mathbf{x} = \langle x_1, x_2, x_3\rangle$ such that

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

so that we must have $x_2 = x_3$ and $x_1 = -x_3$. Hence

$$E_2 = \{\langle -x_3, x_3, x_3 \rangle : x_3 \in \mathbb{R}\}$$

A basis for E_3 is $\{\langle -1, 1, 1\rangle\}$.

10a. Property 1: let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Since $\mathbf{x}^T\mathbf{A}\mathbf{y}$ is formally a 1×1 matrix it must equal its own transpose, and so:

$$\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{x}^T\mathbf{A}\mathbf{y} = (\mathbf{x}^T\mathbf{A}\mathbf{y})^T = \mathbf{y}^T\mathbf{A}^T(\mathbf{x}^T)^T = \mathbf{y}^T\mathbf{A}^T\mathbf{x} = \mathbf{y}^T\mathbf{A}\mathbf{x} = \langle \mathbf{y}, \mathbf{x}\rangle.$$

Property 2: let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$. Then

$$\langle \mathbf{x}, \mathbf{y} + \mathbf{z}\rangle = \mathbf{x}^T\mathbf{A}(\mathbf{y} + \mathbf{z}) = \mathbf{x}^T\mathbf{A}\mathbf{y} + \mathbf{x}^T\mathbf{A}\mathbf{z} = \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{x}, \mathbf{z}\rangle.$$

Property 3: let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$$\langle c\mathbf{x}, \mathbf{y}\rangle = (c\mathbf{x})^T\mathbf{A}\mathbf{y} = c(\mathbf{x}^T\mathbf{A}\mathbf{y}) = c\langle \mathbf{x}, \mathbf{y}\rangle$$

and

$$\langle \mathbf{x}, c\mathbf{y}\rangle = \mathbf{x}^T\mathbf{A}(c\mathbf{y}) = c(\mathbf{x}^T\mathbf{A}\mathbf{y}) = c\langle \mathbf{x}, \mathbf{y}\rangle.$$

10b. Letting

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

we have

$$\left\langle \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = [-2] \to -2 < 0,$$

violating Property 4.