Math 260 Exam #2 Key (Summer 2012)

1. We show that $S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a} \cdot \mathbf{x} \geq c\}$ is convex. For any $\mathbf{u}, \mathbf{v} \in S$ let $\mathbf{x} \in \ell_{\mathbf{uv}}$, the line segment joining $\mathbf{u}$ and $\mathbf{v}$. Then $\mathbf{x} = (1-t)\mathbf{u} + t\mathbf{v}$ for some $t \in [0,1]$. Now, since $\mathbf{u}$ and $\mathbf{v}$ are in S we have $\mathbf{a} \cdot \mathbf{u} \geq c$ and $\mathbf{a} \cdot \mathbf{v} \geq c$, so that

$$\mathbf{a} \cdot \mathbf{x} = (1-t)\mathbf{a} \cdot \mathbf{u} + t\mathbf{a} \cdot \mathbf{v} \geq (1-t)c + tc = c$$

and therefore $\mathbf{x} \in S$. This shows that $\ell_{\mathbf{uv}} \subseteq S$, and since $\mathbf{u}, \mathbf{v} \in S$ are arbitrary it follows that S is convex.

2. The proposition in section 3.4 of the notes (or in section 3.4 of the book) could be used here: $\langle a, b \rangle$ and $\langle c, d \rangle$ are linearly independent if and only if $ad - bc \neq 0$. So, since $(1)(5) - (2)(1) = 3 \neq 0$, we conclude that $\langle 1, 2 \rangle$ and $\langle 1, 5 \rangle$ are linearly independent.

3. Suppose that

$$\sum_{i=1}^{n} c_i \mathbf{v}_i = \mathbf{0}. \tag{1}$$

For any $1 \leq j \leq r$ we have

$$\left(\sum_{i=1}^{n} c_i \mathbf{v}_i \right) \cdot \mathbf{v}_j = \mathbf{0} \cdot \mathbf{v}_j \quad \Rightarrow \quad \sum_{i=1}^{n} c_i (\mathbf{v}_i \cdot \mathbf{v}_j) = 0 \quad \Rightarrow \quad c_j (\mathbf{v}_j \cdot \mathbf{v}_j) = 0 \quad \Rightarrow \quad c_j |\mathbf{v}_j|^2 = 0.$$

Now, since $\mathbf{v}_j \neq \mathbf{0}$ we have $|\mathbf{v}_j| \neq 0$, and so $c_j |\mathbf{v}_j|^2 = 0$ implies $c_j = 0$. We conclude that $c_1 = \cdots = c_r = 0$, which shows that (1) has only the trivial solution and therefore $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are linearly independent.

4. We get a matrix in row-echelon form using elementary row operations on the given matrix:

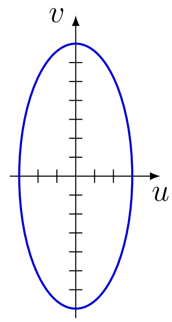
$$\begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 3 \\ 4 & 8 & -12 \\ 1 & -1 & 5 \end{bmatrix} \xrightarrow[-r_1+r_4 \to r_4]{-r_1+r_2 \to r_2,\ -4r_1+r_3 \to r_3} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 8 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_4} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now it is clear that $\text{rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A}) = 2$.

5. T takes $(x, y) \in \mathbb{R}^2$ and returns $(u, v) \in \mathbb{R}^2$ given by $(u, v) = (3x, 7y)$. Thus $x = u/3$ and $y = v/7$, and so from $x^2 + y^2 = 1$ we obtain

$$\left(\frac{u}{3}\right)^2 + \left(\frac{v}{7}\right)^2 = 1.$$

In the uv-coordinate system we find that the image of the circle $x^2 + y^2 = 1$ under T is the ellipse pictured below.



6. It is linear. For $\mathbf{x}_1 = \langle x_1, y_1, z_1 \rangle$ and $\mathbf{x}_2 = \langle x_2, y_2, z_2 \rangle$ in $\mathbb{R}^3$ we have

$$\begin{aligned} T(\mathbf{x}_1 + \mathbf{x}_2) &= T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \langle 2(x_1 + x_2), (y_1 + y_2) - (z_1 + z_2) \rangle \\ &= \langle 2x_1 + 2x_2, (y_1 - z_1) + (y_2 - z_2) \rangle = \langle 2x_1, y_1 - z_1 \rangle + \langle 2x_2, y_2 - z_2 \rangle \\ &= T(\mathbf{x}_1) + T(\mathbf{x}_2). \end{aligned}$$

For $\mathbf{x} = \langle x, y, z \rangle$ and $c \in \mathbb{R}$ we have

$$T(c\mathbf{x}) = T(cx, cy, cz) = \langle 2(cx), cy - cz \rangle = c\langle 2x, y - z \rangle = cT(\mathbf{x}).$$

7. Let C be a convex set in a vector space V, and $L : V \to W$ a linear transformation. We must show that

$$L(C) = \{L(\mathbf{v}) : \mathbf{v} \in V\}$$

is a convex set.

Let $\mathbf{a}, \mathbf{b} \in L(C)$ be arbitrary and fix $\mathbf{x} \in \ell_{\mathbf{ab}}$ (the line segment joining $\mathbf{a}$ and $\mathbf{b}$), so that there exists some $t \in [0, 1]$ such that $\mathbf{x} = (1 - t)\mathbf{a} + t\mathbf{b}$. Now, there exist some $\boldsymbol{\alpha}, \boldsymbol{\beta} \in C$ such that $L(\boldsymbol{\alpha}) = \mathbf{a}$ and $L(\boldsymbol{\beta}) = \mathbf{b}$, and since C is convex the vector $\boldsymbol{\xi} = (1 - t)\boldsymbol{\alpha} + t\boldsymbol{\beta}$ must be an element of C. Observing that

$$L(\boldsymbol{\xi}) = L((1 - t)\boldsymbol{\alpha} + t\boldsymbol{\beta}) = (1 - t)L(\boldsymbol{\alpha}) + tL(\boldsymbol{\beta}) = (1 - t)\mathbf{a} + t\mathbf{b} = \mathbf{x}$$

for $\boldsymbol{\xi} \in C$, we conclude that $\mathbf{x} \in L(C)$. Hence $\ell_{\mathbf{ab}} \subseteq L(C)$ and $L(C)$ is convex.

8a. Suppose that $\mathrm{Ker}(L) = \{\mathbf{0}\}$. Then $\dim(\mathrm{Ker}(L)) = 0$ and by Theorem 3 on the exam we have

$$\dim(V) = \dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = \dim(\mathrm{Im}(L)).$$

Since we're given $\dim(V) = \dim(W)$, it follows that $\dim(\mathrm{Im}(L)) = \dim(W)$. Now, $\mathrm{Im}(L)$ is a subspace of W, and $\dim(\mathrm{Im}(L)) = \dim(W)$ implies that any basis for $\mathrm{Im}(L)$ must also be a basis for W. Therefore $\mathrm{Im}(L) = W$.[1]

8b. Suppose that $\mathrm{Im}(L) = W$. Then by Theorem 3 on the exam we have

$$\dim(V) = \dim(\mathrm{Ker}(L)) + \dim(\mathrm{Im}(L)) = \dim(\mathrm{Ker}(L)) + \dim(W)$$

[1] See Theorem 5.6 in chapter 3 of the book, or the last proposition in section 3.5 of the notes.

$$= \dim(\text{Ker}(L)) + \dim(V),$$

where the last equality makes use of the hypothesis $\dim(V) = \dim(W)$. Subtracting $\dim(V)$ from both sides of $\dim(V) = \dim(\text{Ker}(L)) + \dim(V)$ then yields $\dim(\text{Ker}(L)) = 0$, which in turn implies that $\text{Ker}(L) = \{\mathbf{0}\}$.

9. An equivalent statement is the contrapositive: "If $\text{Ker}(L) = \{\mathbf{0}\}$, then $\dim(V) \le \dim(W)$." So, suppose that $\text{Ker}(L) = \{\mathbf{0}\}$. Then $\dim(\text{Ker}(L)) = 0$, and by Theorem 3 on the exam we have $\dim(V) = \dim(\text{Im}(L))$. Now, $\text{Im}(L)$ is a subspace of W, so by Theorem 2 on the exam we have $\dim(\text{Im}(L)) \le \dim(W)$. Therefore $\dim(V) \le \dim(W)$.

10. The system can be cast in the form of a matrix equation $\mathbf{Ax} = \mathbf{0}$, where

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -1 \\ 2 & 1 & 1 \end{bmatrix}$$

If S is the solution space of the system, then by an established theorem we have $\dim(S) = \dim(\mathbb{R}^3) - \text{rank}(\mathbf{A}) = 3 - \text{rank}(\mathbf{A})$. Now, the column vectors

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

of $\mathbf{A}$ are linearly independent, so $\text{rank}(\mathbf{A}) \ge 2$. On the other hand $\mathbf{A}$ has only two row vectors, so $\text{rank}(\mathbf{A}) \le 2$. Therefore $\text{rank}(\mathbf{A}) = 2$ and we obtain $\dim(S) = 1$.

Now to get a basis for S. From the first equation in the system we have $z = 2x + y$, which when put into the second equation yields $2x + y + (2x + y) = 0$ and then $2x + y = 0$. Hence $z = 0$ and $y = -2x$. This gives us S itself:

$$S = \{\langle x, y, z\rangle : x \in \mathbb{R},\ y = -2x,\ z = 0\} = \{\langle x, -2x, 0\rangle : x \in \mathbb{R}\} = \{x\langle 1, -2, 0\rangle : x \in \mathbb{R}\}.$$

We can see that $\text{Span}\{\langle 1, -2, 0\rangle\} = S$, and so $\{\langle 1, -2, 0\rangle\}$ is a basis for S.

11. Let $\mathbf{r}_1 = \langle 1, 1, -2, 3, 4, 5, 6\rangle$ and $\mathbf{r}_2 = \langle 0, 0, 2, 1, 0, 7, 0\rangle$. The subspace of $\mathbb{R}^7$ that is orthogonal to both $\mathbf{r}_1$ and $\mathbf{r}_2$ is

$$S = \{\mathbf{x} \in \mathbb{R}^7 : \mathbf{r}_1 \cdot \mathbf{x} = 0 \text{ and } \mathbf{r}_2 \cdot \mathbf{x} = 0\}.$$

Indeed, if we define

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & -2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 2 & 1 & 0 & 7 & 0 \end{bmatrix}$$

then we find that

$$S = \{\mathbf{x} \in \mathbb{R}^7 : \mathbf{Ax} = \mathbf{0}\} = \text{Nul}(\mathbf{A}).$$

By an established theorem we have $\dim(S) = \dim(\mathbb{R}^7) - \text{rank}(\mathbf{A}) = 7 - \text{rank}(\mathbf{A})$. Now, $\mathbf{A}$ is already in row-echelon form, and so it should be clear that the row vectors of $\mathbf{A}$, which are $\mathbf{r}_1$ and $\mathbf{r}_2$, are linearly independent. Thus $\text{rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A}) = 2$, and therefore $\dim(S) = 7 - 2 = 5$.

12. S is an arbitrary line in V, which is to say that $S = \{\mathbf{u} + t\mathbf{v} : t \in \mathbb{R}\}$ for some $\mathbf{u}, \mathbf{v} \in V$. For $L : V \to W$ there are four cases: (1) $L(\mathbf{u}) = L(\mathbf{v}) = \mathbf{0}$; (2) $L(\mathbf{u}) = \mathbf{0}$ and $L(\mathbf{v}) \neq \mathbf{0}$; (3) $L(\mathbf{u}) \neq \mathbf{0}$ and $L(\mathbf{v}) = \mathbf{0}$; and (4) $L(\mathbf{u}), L(\mathbf{v}) \neq \mathbf{0}$. Since $L(\mathbf{u} + t\mathbf{v}) = L(\mathbf{u}) + tL(\mathbf{v})$, we find that cases (1) and (3) result in a point in W, and cases (2) and (4) result in a line in W.

13. By definition $U^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{u} = 0 \text{ for all } \mathbf{u} \in U\}$. Certainly $\mathbf{0} \in U^\perp$. It remains to verify that $U^\perp$ is closed under vector addition and scalar multiplication.

Suppose that $\mathbf{x}, \mathbf{y} \in U^\perp$, so that $\mathbf{x} \cdot \mathbf{u} = 0$ and $\mathbf{y} \cdot \mathbf{u} = 0$ for all $\mathbf{u} \in U$. Since for all $\mathbf{u} \in U$ we have

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} + \mathbf{y} \cdot \mathbf{u} = 0 + 0 = 0,$$

it follows that $\mathbf{x} + \mathbf{y} \in U^\perp$.

Now suppose that $\mathbf{x} \in U^\perp$ and $c \in \mathbb{R}$. Then for any $\mathbf{u} \in U$ we have

$$(c\mathbf{x}) \cdot \mathbf{u} = c(\mathbf{x} \cdot \mathbf{u}) = c(0) = 0,$$

which implies that $c\mathbf{x} \in U^\perp$.

Therefore $U^\perp$ is a subspace of $\mathbb{R}^n$.