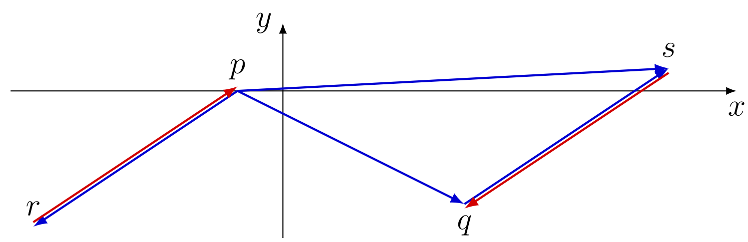
Math 260 Exam #1 Key (Summer 2012)

1. Quite trivially we have

$$\begin{aligned} 3p &= 3(3,-1,8) = (9,-3,24), \\ p+q &= (3,-1,8)+(-2,-9,0) = (1,-10,8), \\ p-2q &= (3,-1,8)-2(-2,-9,0) = (7,17,8). \end{aligned}$$

2. We have $\overrightarrow{rp} \sim \overrightarrow{qs}$ since $p - r = (-2,0) - (-11,-6) = (9,6) = (17,1) - (8,-5) = s - q$. Also $\overrightarrow{pr} \sim \overrightarrow{sq}$.



3. Since $\mathbf{u} \in \mathbb{R}^n$ we have $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$. For each $1 \le i \le n$ let $\mathbf{e}_i = \langle 0, \ldots, 1, \ldots, 0 \rangle$, the vector with ith coordinate 1 and all other coordinates 0. Since $\mathbf{u}$ is orthogonal to all vectors in $\mathbb{R}^n$, we have

$$u_i = (u_1)(0) + (u_2)(0) + \cdots + (u_i)(1) + \cdots + (u_n)(0) = \mathbf{u} \cdot \mathbf{e}_i = 0$$

for all i. Thus $\mathbf{u} = \mathbf{0}$.

4a. $|\mathbf{u}| = \sqrt{2^2 + (-1)^2 + 5^2} = \sqrt{30}$ and $|\mathbf{v}| = \sqrt{(-1)^2 + 1^2 + 1^2} = \sqrt{3}$.

4b. Since $\mathbf{u} \cdot \mathbf{v} = (2)(-1) + (-1)(1) + (5)(1) = 2$ and $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2 = (\sqrt{3})^2 = 3$, we have

$$\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\frac{\mathbf{u} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \right) \mathbf{v} = \frac{2}{3} \langle -1, 1, 1 \rangle = \left\langle -\frac{2}{3}, \frac{2}{3}, \frac{2}{3} \right\rangle.$$

4c. We have

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{2}{\sqrt{30}\sqrt{3}} = \frac{2}{3\sqrt{10}} \quad \Rightarrow \quad \theta = \cos^{-1}\left(\frac{2}{3\sqrt{10}} \right) \approx 77.8^\circ.$$

5a. The easiest 10 points you could hope for:

$$\mathbf{A}^T = \begin{bmatrix} 2 & -1 \\ -3 & 4 \\ 0 & 6 \end{bmatrix}$$

5b. We have

$$\mathbf{AB} = \begin{bmatrix} -14 & 7 \\ 25 & -55 \end{bmatrix}$$

6. We have $\mathbf{A}^2 = \mathbf{AA}$, $\mathbf{A}^3 = \mathbf{AA}^2$, $\mathbf{A}^4 = \mathbf{AA}^3$, so

$$\mathbf{A}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}, \quad \mathbf{A}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 81 \end{bmatrix}.$$

7. We must find a 2×2 matrix with entries a, b, c, and d such that

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a\cos\theta - c\sin\theta & b\cos\theta - d\sin\theta \\ a\sin\theta + c\cos\theta & b\sin\theta + d\cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

To get $a\cos\theta - c\sin\theta = 1$, let $a = \cos\theta$ and $c = -\sin\theta$; to get $b\cos\theta - d\sin\theta = 0$ let $b = \sin\theta$ and $d = \cos\theta$. It can be seen that these choices work, and moreover

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

also holds for these choices. Thus, the inverse of $\mathbf{R}_\theta$ is

$$\mathbf{R}_\theta^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

8. From the second equation we have $y = -x$. Putting this into the first and third equations gives $11x + 3z = 0$ and $x + 6z = 0$. From the latter equation comes $x = -6z$, which when put into the former equation yields $11(-6z) + 3z = 0$, and thus $z = 0$. Since $z = 0$, we get $x = -6(0) = 0$, and then $y = -0 = 0$. That is, $x = y = z = 0$ is the only solution.

9. Call the matrix $\mathbf{A}$. Then,

$$\mathbf{A} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \end{bmatrix} \xrightarrow{-r_1 + r_3 \to r_3} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \end{bmatrix} \xrightarrow{-2r_2 + r_3 \to r_3} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

10. Letting $\mathbf{A}$ be the matrix, we have

$$\left[\begin{array}{ccc|ccc} 2 & 4 & 3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-r_1 \to r_1]{r_2 \leftrightarrow r_1} \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2r_1 + r_2 \to r_2} \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 10 & 3 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 10 & 3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{-5r_2 + r_3 \to r_3} \left[\begin{array}{ccc|ccc} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow{\frac{1}{2}r_2 \to r_2}$$

$$\left[\begin{array}{ccc|ccc}1 & -3 & 0 & 0 & -1 & 0\\ 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2}\\ 0 & 0 & -2 & 1 & 2 & -5\end{array}\right] \xrightarrow{3r_2+r_1\to r_1} \left[\begin{array}{ccc|ccc}1 & 0 & \frac{3}{2} & 0 & -1 & \frac{3}{2}\\ 0 & 1 & \frac{1}{2} & 0 & 0 & \frac{1}{2}\\ 0 & 0 & -2 & 1 & 2 & -5\end{array}\right] \xrightarrow[\frac{3}{4}r_3+r_1\to r_1]{\frac{1}{4}r_3+r_2\to r_2}$$

$$\left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & -\frac{9}{4}\\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & -\frac{3}{4}\\ 0 & 0 & -2 & 1 & 2 & -5\end{array}\right] \xrightarrow{-\frac{1}{2}r_3\to r_3} \left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{3}{4} & \frac{1}{2} & -\frac{9}{4}\\ 0 & 1 & 0 & \frac{1}{4} & \frac{1}{2} & -\frac{3}{4}\\ 0 & 0 & 1 & -\frac{1}{2} & -1 & \frac{5}{2}\end{array}\right].$$

Thus

$$\mathbf{A}^{-1} = \begin{bmatrix}\frac{3}{4} & \frac{1}{2} & -\frac{9}{4}\\ \frac{1}{4} & \frac{1}{2} & -\frac{3}{4}\\ -\frac{1}{2} & -1 & \frac{5}{2}\end{bmatrix}$$

11. V is not a vector space: it lacks $\mathbf{O}$, so A3 fails. Define matrices

$$\mathbf{A} = \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} \quad\text{and}\quad \mathbf{B} = \begin{bmatrix}2 & 0\\ 1 & 3\end{bmatrix},$$

both of which are in V. A4 and the closure property of scalar multiplication fail since $-\mathbf{A}$ is not in V. The closure property of addition fails since $\mathbf{A}+\mathbf{B} \notin V$.

12. Let $W = \{(x,y) : x + 4y = 0\}$. We show that W is a subspace of $\mathbb{R}^2$. Since $0 + 4(0) = 0$, we have $(0,0) \in W$. Next, if $(a,b),(c,d) \in W$ so that $a + 4b = 0$ and $c + 4d = 0$, it's clear to see that $(a+c, b+d) \in W$ since

$$(a+c) + 4(b+d) = (a+4b) + (c+4d) = 0 + 0 = 0.$$

Finally, $s(a,b) \in W$ for any $s \in \mathbb{R}$ since $s(a,b) = (sa, sb)$, and $sa + 4(sb) = s(a+4b) = s(0) = 0$. Therefore W is a subspace.

13. By definition $U + W = \{\mathbf{u} + \mathbf{w} : \mathbf{u} \in U \text{ and } \mathbf{w} \in W\}$. Since U and W are subspaces we have $\mathbf{0} \in U$ and $\mathbf{0} \in W$, and therefore $\mathbf{0} = \mathbf{0} + \mathbf{0} \in U + W$.

Let $a \in \mathbb{R}$, and suppose that $\mathbf{x} \in U + W$. Then $\mathbf{x} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u} \in U$ and $\mathbf{w} \in W$, and since U and W are subspaces we also have $a\mathbf{u} \in U$ and $a\mathbf{w} \in W$. Now we find that $a\mathbf{x} \in U + W$ since $a\mathbf{x} = a(\mathbf{u} + \mathbf{w}) = a\mathbf{u} + a\mathbf{w}$.

Finally, let $\mathbf{x}, \mathbf{y} \in U + W$. Then $\mathbf{x} = \mathbf{u}_1 + \mathbf{w}_1$ and $\mathbf{y} = \mathbf{u}_2 + \mathbf{w}_2$ for some $\mathbf{u}_1, \mathbf{u}_2 \in U$ and $\mathbf{w}_1, \mathbf{w}_2 \in W$. Now

$$\mathbf{x} + \mathbf{y} = (\mathbf{u}_1 + \mathbf{w}_1) + (\mathbf{u}_2 + \mathbf{w}_2) = (\mathbf{u}_1 + \mathbf{u}_2) + (\mathbf{w}_1 + \mathbf{w}_2),$$

and since $\mathbf{u}_1 + \mathbf{u}_2 \in U$ and $\mathbf{w}_1 + \mathbf{w}_2 \in W$ it follows that $\mathbf{x} + \mathbf{y} \in U + W$.

Therefore $U + W$ is a subspace of V.

14a. Let $\langle a, b, c\rangle \in \mathbb{R}^3$. We attempt to find scalars $x, y, z \in \mathbb{R}$ such that

$$x\langle 1,1,1\rangle + y\langle 2,2,0\rangle + z\langle 3,0,0\rangle = \langle a,b,c\rangle.$$

This yields the system

$$\begin{cases} x + 2y + 3z = a \\ x + 2y \qquad\;\; = b \\ x \qquad\qquad\quad\; = c \end{cases}$$

which indeed has a solution:

$$(x, y, z) = \left(c, \frac{b-c}{2}, \frac{a-b}{3} \right).$$

Thus every vector in $\mathbb{R}^3$ is expressible as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, which shows that the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$.

14b. Let $\langle a, b, c\rangle \in \mathbb{R}^3$. We attempt to find scalars $x, y, z \in \mathbb{R}$ such that

$$x\langle 2, -1, 3\rangle + y\langle 4, 1, 2\rangle + z\langle 8, -1, 8\rangle = \langle a, b, c\rangle.$$

This yields the system

$$\begin{cases} 2x + 4y + 8z = a \\ -x + \;\; y - \;\; z = b \\ 3x + 2y + 8z = c \end{cases}$$

This can be cast as an augmented matrix and manipulated using elementary row operations:

$$\left[\begin{array}{rrr|c} 2 & 4 & 8 & a \\ -1 & 1 & -1 & b \\ 3 & 2 & 8 & c \end{array}\right] \sim \left[\begin{array}{rrr|c} -1 & 1 & -1 & b \\ 2 & 4 & 8 & a \\ 3 & 2 & 8 & c \end{array}\right] \sim \left[\begin{array}{rrr|c} -1 & 1 & -1 & b \\ 0 & 6 & 6 & 2b+a \\ 0 & 5 & 5 & 3b+c \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|c} 1 & -1 & 1 & -b \\ 0 & 1 & 1 & \frac{2b+a}{6} \\ 0 & 5 & 5 & 3b+c \end{array}\right] \sim \left[\begin{array}{rrr|c} 1 & -1 & 1 & -b \\ 0 & 1 & 1 & \frac{2b+a}{6} \\ 0 & 0 & 0 & 3b+c-5\left(\frac{2b+a}{6}\right) \end{array}\right]$$

We see that in order for $\langle a, b, c\rangle$ to be expressed as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, we need a, b, and c such that

$$3b + c - 5\left(\frac{2b+a}{6}\right) = 0,$$

or

$$-\frac{5}{6}a + \frac{4}{3}b + c = 0.$$

This leads to $1 = 0$ if we choose $\langle a, b, c\rangle$ to be $\langle 0, 0, 1\rangle$, for instance. That is, we cannot express $\langle 0, 0, 1\rangle$ (among many other vectors) as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$. We conclude that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ does not span $\mathbb{R}^3$.