# Math 260 Exam #5 Key (Spring 2014)

**1a** Find the characteristic polynomial:

$$P_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}) = \begin{vmatrix} 2-t & 0 & -2 \\ 0 & 3-t & 0 \\ 0 & 0 & 3-t \end{vmatrix} = (2-t)\begin{vmatrix} 3-t & 0 \\ 0 & 3-t \end{vmatrix} = (2-t)(3-t)^2.$$

The characteristic equation is $(2-t)(3-t)^2 = 0$, which has solution set $\{2, 3\}$. Hence the eigenvalues of $\mathbf{A}$ are 2 and 3.

**1b** The eigenspace corresponding to 2 is

$$E_{\mathbf{A}}(2) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{A}\mathbf{x} = 2\mathbf{x}\} = \{\mathbf{x} : (\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0}\}.$$

Passing to the augmented matrix for the system $(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \mathbf{0}$, we have

$$\left[\begin{array}{ccc|c} 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Thus the solution set of the system is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : y = z = 0 \text{ and } x \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

A basis for $E_{\mathbf{A}}(2)$ is thus $\mathcal{B}_1 = \{[1, 0, 0]^\top\}$.

The eigenspace corresponding to 3 is

$$E_{\mathbf{A}}(3) = \{\mathbf{x} \in \mathbb{R}^3 : \mathbf{A}\mathbf{x} = 3\mathbf{x}\} = \{\mathbf{x} : (\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}\}.$$

Passing to the augmented matrix for the system $(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \mathbf{0}$, we have

$$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the solution set of the system is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x = -2z \text{ and } y, z \in \mathbb{R} \right\} = \left\{ s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\} = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for $E_{\mathbf{A}}(3)$ is thus $\mathcal{B}_2 = \{[0, 1, 0]^\top, [-2, 0, 1]^\top\}$.

**1c** A spectral basis for $\mathbf{A}$ (i.e. a basis for $\mathbb{R}^3$ consisting of linearly independent eigenvectors of $\mathbf{A}$) is the ordered basis

$$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 = \left( \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right).$$

The eigenvalues corresponding to these eigenvalues are 2, 3, and 3, respectively. Therefore the diagonal matrix we seek is

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

As for $\mathbf{P}$, that is the $3 \times 3$ matrix with column vectors being the vectors in $\mathcal{B}$ in the order that they appear:

$$\mathbf{P} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**2** By Ye Olde Gram-Schmidt Process,

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} - \frac{5}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix},$$

and

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \begin{bmatrix} 2 \\ 1 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - 3\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix}.$$

An orthogonal basis for $W$ is therefore

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}.$$

Next, find the norms of the vectors $\mathbf{w}_1$, $\mathbf{w}_2$, and $\mathbf{w}_3$ found above:

$$\|\mathbf{w}_1\| = \sqrt{2}, \quad \|\mathbf{w}_2\| = \frac{1}{\sqrt{2}}, \quad \|\mathbf{w}_3\| = \sqrt{10}.$$

An orthonormal basis for $W$ is thus

$$\left\{ \frac{\mathbf{w}_1}{\|\mathbf{w}_1\|}, \frac{\mathbf{w}_2}{\|\mathbf{w}_2\|}, \frac{\mathbf{w}_3}{\|\mathbf{w}_3\|} \right\} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \sqrt{2}\begin{bmatrix} 1/2 \\ 0 \\ -1/2 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} \right\}.$$