Math 260 Exam #4 Key (Spring 2014)

1a Since $5-2(7)+3(3)=0$, we see that $(x,y,z)=(5,7,3)$ is a solution to $x-2y+3z=0$, and hence $\mathbf{v}\in W$. To find the $\mathcal{B}$-coordinates of $\mathbf{v}$, we find $a,b\in\mathbb{R}$ such that

$$a\begin{bmatrix}-1\\1\\1\end{bmatrix}+b\begin{bmatrix}1\\2\\1\end{bmatrix}=\begin{bmatrix}5\\7\\3\end{bmatrix},$$

which is to say we solve the system

$$\begin{cases}-a+\ \ b=5\\ \ \ \ a+2b=7\\ \ \ \ a+\ \ b=3\end{cases}$$

The only solution is $(a,b)=(-1,4)$, and therefore

$$[\mathbf{v}]_{\mathcal{B}}=\begin{bmatrix}-1\\4\end{bmatrix}.$$

1b Letting $\mathbf{u}_1=[-1\ 1\ 1]^\top$ and $\mathbf{u}_2=[1\ 2\ 1]^\top$, by an established theorem we have

$$\mathbf{M}=\Big[[\mathbf{u}_1]_{\mathcal{C}}\ \ [\mathbf{u}_2]_{\mathcal{C}}\Big],$$

and so we must find the $\mathcal{C}$-coordinates of $\mathbf{u}_1$ and $\mathbf{u}_2$. Starting with $\mathbf{u}_1$, we find $a,b\in\mathbb{R}$ such that $a\mathbf{v}_1+b\mathbf{v}_2=\mathbf{u}_1$; that is,

$$a\begin{bmatrix}2\\1\\0\end{bmatrix}+b\begin{bmatrix}-3\\0\\1\end{bmatrix}=\begin{bmatrix}-1\\1\\1\end{bmatrix},$$

which has $(a,b)=(1,1)$ as the only solution, and hence

$$[\mathbf{u}_1]_{\mathcal{C}}=\begin{bmatrix}1\\1\end{bmatrix}.$$

Next, we find $a,b\in\mathbb{R}$ such that $a\mathbf{v}_1+b\mathbf{v}_2=\mathbf{u}_2$; that is,

$$a\begin{bmatrix}2\\1\\0\end{bmatrix}+b\begin{bmatrix}-3\\0\\1\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix},$$

which has $(a,b)=(2,1)$ as the only solution, and hence

$$[\mathbf{u}_2]_{\mathcal{C}}=\begin{bmatrix}2\\1\end{bmatrix}.$$

Therefore

$$\mathbf{M}=\begin{bmatrix}1&2\\1&1\end{bmatrix}.$$

1c Using the $\mathcal{B}$-coordinates of $\mathbf{v}$ found above, we have

$$[\mathbf{v}]_{\mathcal{C}}=\mathbf{M}[\mathbf{v}]_{\mathcal{B}}=\begin{bmatrix}1&2\\1&1\end{bmatrix}\begin{bmatrix}-1\\4\end{bmatrix}=\begin{bmatrix}7\\3\end{bmatrix}.$$

2 We have $\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2)$, and $\mathcal{C} = (\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3)$. Now,

$$L(\mathbf{v}_1) = L\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -5(3) + 13(1) \\ -7(3) + 16(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}.$$

We need the $\mathcal{C}$-coordinates of $L(\mathbf{v}_1)$, which means finding a_1, a_2, a_3 such that

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = L(\mathbf{v}_1);$$

that is,

$$\begin{cases} a_1 - a_2 = 1 \\ 2a_2 + a_3 = -2 \\ -a_1 + 2a_2 + 2a_3 = -5, \end{cases}$$

which solves to give $a_1 = 1$, $a_2 = 0$, and $a_3 = -2$. Thus

$$[L(\mathbf{v}_1)]_{\mathcal{C}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}.$$

Next,

$$L(\mathbf{v}_2) = L\left(\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}.$$

We need the $\mathcal{C}$-coordinates of $L(\mathbf{v}_2)$, so we find a_1, a_2, a_3 such that

$$a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = a_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + a_2\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} + a_3\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = L(\mathbf{v}_2).$$

Like before, this yields a system of equations. We put its augmented matrix into row-echelon form:

$$\left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ -1 & 2 & 2 & -3 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 0 & 2 & 1 & 1 \\ 0 & 1 & 2 & -1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|r} 1 & -1 & 0 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -3 & 3 \end{array}\right]$$

which solves to give $a_1 = 3$, $a_2 = 1$, and $a_3 = -1$. Thus

$$[L(\mathbf{v}_2)]_{\mathcal{C}} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

The $\mathcal{BC}$-matrix of L is therefore

$$[L]_{\mathcal{BC}} = \Big[\big[L(\mathbf{v}_1)\big]_{\mathcal{C}} \ \ \big[L(\mathbf{v}_2)\big]_{\mathcal{C}}\Big] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}.$$

3a Find the characteristic polynomial:

$$P_{\mathbf{A}}(t) = \det(\mathbf{A} - t\mathbf{I}) = \begin{vmatrix} -t & 0 & 2 & 0 \\ 1 & -t & 1 & 0 \\ 0 & 1 & -2-t & 0 \\ 0 & 0 & 0 & 1-t \end{vmatrix} = (1-t)\begin{vmatrix} -t & 0 & 2 \\ 1 & -t & 1 \\ 0 & 1 & -2-t \end{vmatrix}$$

$$= (1-t)\left[-t\begin{vmatrix} -t & 1 \\ 1 & -2-t \end{vmatrix} + 2\begin{vmatrix} 1 & -t \\ 0 & 1 \end{vmatrix}\right] = (1-t)(t^3 + 2t^2 - t - 2)$$

$$= (t-1)[t^2(t+2) - (t+2)] = (t-1)(t+2)(t^2-1) = (t-1)^2(t+1)(t+2).$$

3b From above, we see that $P_{\mathbf{A}}(t) = 0$ if and only if $t = -2, -1, 1$, and so these are the eigenvalues of $\mathbf{A}$.

3c The smallest eigenvalue is -2. The corresponding eigenspace is

$$E_{\mathbf{A}}(-2) = \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{A}\mathbf{x} = -2\mathbf{x}\}.$$

Now, $\mathbf{A}\mathbf{x} = -2\mathbf{x}$ is equivalent to $(\mathbf{A}+2\mathbf{I})\mathbf{x} = \mathbf{0}$, which is a system of equations with augmented matrix

$$\left[\begin{array}{cccc|c} 2 & 0 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right]$$

From this we find that $x_4 = 0$, $x_2 = 0$, and $x_1 = -x_3$; that is, $\mathbf{x} \in E_{\mathbf{A}}(-2)$ if and only if

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_3$$

for some $x_3 \in \mathbb{R}$. Replacing x_3 with t, we therefore have

$$E_{\mathbf{A}}(-2) = \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} t : t \in \mathbb{R} \right\}.$$

A basis for $E_{\mathbf{A}}(-2)$ is thus $\{[-1, 0, 1, 0]^\top\}$.

3d The largest eigenvalue is 1. The corresponding eigenspace is

$$E_{\mathbf{A}}(1) = \{\mathbf{x} \in \mathbb{R}^4 : \mathbf{A}\mathbf{x} = \mathbf{x}\}.$$

Now, $\mathbf{A}\mathbf{x} = \mathbf{x}$ is equivalent to $(\mathbf{A} - \mathbf{I})\mathbf{x} = \mathbf{0}$, which is a system of equations with augmented matrix

$$\left[\begin{array}{rrrr|r} -1 & 0 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrr|r} -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The last augmented matrix gives the system

$$\begin{cases} -x_1 + 2x_3 = 0 \\ -x_2 + 3x_3 = 0 \end{cases}$$

Thus $x_1 = 2x_3$ and $x_2 = 3x_3$, where x_3 and x_4 are free. Replacing x_3 and x_4 with s and t, respectively, we have

$$E_{\mathbf{A}}(1) = \left\{ \begin{bmatrix} 2s \\ 3s \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} t : s, t \in \mathbb{R} \right\}$$

A basis for $E_{\mathbf{A}}(1)$ is thus

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$