Math 260 Exam #3 Key (Spring 2014)

1 Two vectors belonging to U are

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

However the vector $\mathbf{u}+\mathbf{v}$ does not belong to U. Since U is not closed under vector addition, it is not a subspace of $\mathbb{R}^3$.

2a Suppose $\mathbf{x}, \mathbf{y} \in W_1 \cap W_2$ and c is a scalar. Since $\mathbf{x}, \mathbf{y} \in W_1$ and W_1 is a subspace, we have $c\mathbf{x} \in W_1$ and $\mathbf{x}+\mathbf{y} \in W_1$. Since $\mathbf{x}, \mathbf{y} \in W_2$ and W_2 is a subspace, we have $c\mathbf{x} \in W_2$ and $\mathbf{x}+\mathbf{y} \in W_2$. Therefore $c\mathbf{x} \in W_1 \cap W_2$ and $\mathbf{x}+\mathbf{y} \in W_1 \cap W_2$. We have now shown that $W_1 \cap W_2$ is closed under scalar multiplication and vector addition, and therefore $W_1 \cap W_2$ is a subspace.

2b To see that $W_1 \cup W_2$ is not necessarily a subspace of V, we consider a counterexample. Let $V = \mathbb{R}^2$, and define subspaces $W_1, W_2 \subseteq \mathbb{R}^2$ thus:

$$W_1 = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{bmatrix} 0 \\ y \end{bmatrix} : y \in \mathbb{R} \right\}.$$

Let

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Since $\mathbf{u} \in W_1$ and $\mathbf{v} \in W_2$, we have $\mathbf{u}, \mathbf{v} \in W_1 \cup W_2$. But:

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \notin W_1 \cup W_2,$$

so that $W_1 \cup W_2$ is seen to not be closed under vector addition, and therefore $W_1 \cup W_2$ is not a subspace of $\mathbb{R}^2$.

However, in general, if we're given either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, then $W_1 \cup W_2$ equals either W_2 or W_1, respectively, and therefore $W_1 \cup W_2$ is a subspace of V.

3 Let

$$\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

and find $x, y, z, w \in \mathbb{R}$ such that

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} z + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} w = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

which gives rise to a system of equation with augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 1 & 0 & 1 & a \\ 2 & 0 & 1 & 1 & b \\ 3 & -1 & 1 & 0 & c \end{array}\right] \sim \left[\begin{array}{cccc|l} 1 & 1 & 0 & 1 & a \\ 0 & 2 & -1 & 1 & 2a-b \\ 0 & 0 & 1 & 1 & 2b-a-c \end{array}\right]$$

From this we get

$$x = \frac{a-b+c}{2}, \quad y = \frac{a+b-c}{2} - w, \quad z = 2b - a - c - w,$$

where w may be any real number. This shows that S spans $\mathbb{R}^3$, and moreover any $\mathbf{v} \in \mathbb{R}^3$ can be written as a linear combination of vectors in S in infinitely many ways (one for each value of w).

ANOTHER METHOD: let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_4 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 0 \end{bmatrix}.$$

We have $\det(\mathbf{A}) = -2 \neq 0$, so the Invertible Matrix Theorem implies that $\operatorname{rank}(\mathbf{A}) = 3$, which is to say the column vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of $\mathbf{A}$ are linearly independent and therefore form a basis for $\mathbb{R}^3$. Since $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ contains the column vectors of $\mathbf{A}$, it immediately follows that $\operatorname{Span}(S) = \mathbb{R}^3$. Now, for any $a \in \mathbb{R}$, since

$$\mathbf{v} - a\mathbf{u}_4 \in \mathbb{R}^3 \quad \text{and} \quad \operatorname{Span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\} = \mathbb{R}^3,$$

there exist $c_1, c_2, c_3 \in \mathbb{R}$ such that

$$c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 = \mathbf{v} - a\mathbf{u}_4,$$

and therefore

$$\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + a\mathbf{u}_4.$$

Of course a is arbitrary, which shows that $\mathbf{v}$ can be written as a linear combination of vectors in S in infinitely many ways.

4 We can show the vectors are linearly independent by showing they form a basis for $\mathbb{R}^3$. Define the matrix $\mathbf{A} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \mathbf{x}_3]$, which is the matrix with $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ as its column vectors. In the textbook there is a theorem that implies that the column vectors of $\mathbf{A}$ are a basis for $\mathbb{R}^3$ if and only if $\det(\mathbf{A}) \neq 0$. We have

$$\det(\mathbf{A}) = \begin{vmatrix} 2 & 3 & -2 \\ 1 & 4 & 3 \\ -1 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -5 & -8 \\ 1 & 4 & 3 \\ 0 & 10 & 5 \end{vmatrix} = (-1)^{2+1}(1) \begin{vmatrix} -5 & -8 \\ 10 & 5 \end{vmatrix} = -55 \neq 0,$$

and so $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are linearly independent.

Solving a system, we find that

$$-\frac{122}{55}\mathbf{x}_1 + \frac{38}{55}\mathbf{x}_2 - \frac{2}{11}\mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 6 \end{bmatrix}$$

5 We have $z = 2y - x$, so the plane consists of points (x, y, z) such that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 2y - x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} y.$$

It is clear that the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

correspond to points that lie on the plane, and moreover they are linearly independent since the matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$$

has rank 2. (The first two row vectors are linearly independent, so the matrix has rank at least 2; but the rank cannot be greater than 2 since the matrix has only two column vectors.) Since the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans the plane (the set $\{s\mathbf{v}_1 + t\mathbf{v}_2 : s, t \in \mathbb{R}\}$ is precisely the solution set of the equation $x - 2y + z = 0$), we conclude that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for the plane.