Math 260 Exam #2 Key (Spring 2014)

1 We employ the same sequence of elementary row operations on both $\mathbf{A}$ and $\mathbf{I}_3$, as follows.

$$\left[\begin{array}{rrr|rrr} 2 & 4 & 3 & 1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[-r_1\to r_1]{r_2\leftrightarrow r_1} \left[\begin{array}{rrr|rrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 2 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-2r_1+r_2\to r_2}$$

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 10 & 3 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2\leftrightarrow r_3} \left[\begin{array}{rrr|rrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 10 & 3 & 1 & 2 & 0 \end{array}\right] \xrightarrow{-5r_2+r_3\to r_3}$$

$$\left[\begin{array}{rrr|rrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow{\frac{1}{2}r_2\to r_2} \left[\begin{array}{rrr|rrr} 1 & -3 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow{3r_2+r_1\to r_1}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 3/2 & 0 & -1 & 3/2 \\ 0 & 1 & 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow[\frac{3}{4}r_3+r_1\to r_1]{\frac{1}{4}r_3+r_2\to r_2} \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3/4 & 1/2 & -9/4 \\ 0 & 1 & 0 & 1/4 & 1/2 & -3/4 \\ 0 & 0 & -2 & 1 & 2 & -5 \end{array}\right] \xrightarrow{-\frac{1}{2}r_3\to r_3}$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 3/4 & 1/2 & -9/4 \\ 0 & 1 & 0 & 1/4 & 1/2 & -3/4 \\ 0 & 0 & 1 & -1/2 & -1 & 5/2 \end{array}\right].$$

Therefore

$$\mathbf{A}^{-1} = \begin{bmatrix} 3/4 & 1/2 & -9/4 \\ 1/4 & 1/2 & -3/4 \\ -1/2 & -1 & 5/2 \end{bmatrix}.$$

To solve $\mathbf{Ax} = \mathbf{b}$, the idea is to simply use $\mathbf{A}^{-1}$:

$$\mathbf{Ax} = \mathbf{b} \quad\Rightarrow\quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \frac{1}{4}\begin{bmatrix} 3 & 2 & -9 \\ 1 & 2 & -3 \\ -2 & -4 & 10 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -15 \\ -5 \\ 18 \end{bmatrix} = \begin{bmatrix} -15/4 \\ -5/4 \\ 9/2 \end{bmatrix}.$$

2 Let $\mathbf{C}$ denote the matrix. Then

$$\det(\mathbf{C}) \xlongequal[-2r_1+r_4\to r_4]{\substack{-2r_1+r_2\to r_2 \\ -r_1+r_3\to r_3}} \begin{vmatrix} 1 & -4 & 3 & 2 \\ 0 & 1 & -1 & -3 \\ 0 & 6 & 3 & -2 \\ 0 & -2 & 8 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 & -3 \\ 6 & 3 & -2 \\ -2 & 8 & 0 \end{vmatrix}$$

$$\xlongequal{4c_1+c_2\to c_2} \begin{vmatrix} 1 & 3 & -3 \\ 6 & 27 & -2 \\ -2 & 0 & 0 \end{vmatrix} = (-1)^{3+1}(-2)\begin{vmatrix} 3 & -3 \\ 27 & -2 \end{vmatrix}$$

$$= -2[(3)(-2) - (-3)(27)] = -150$$

3 Here $\mathbf{A}\mathbf{x} = \mathbf{b}$ with

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 4 & -3 & 1 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

We have

$$\det(\mathbf{A}) = 2\begin{vmatrix} 3 & -2 \\ -3 & 1 \end{vmatrix} - \begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix} = 2(-3) - 2 + 4(-1) = -12,$$

and by Cramer's Rule,

$$x = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 1 & -1 & 1 \\ 0 & 3 & -2 \\ 2 & -3 & 1 \end{vmatrix} = -\frac{1}{12}(-5) = \frac{5}{12}$$

$$y = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 2 & 1 & 1 \\ 1 & 0 & -2 \\ 4 & 2 & 1 \end{vmatrix} = -\frac{1}{12}(1) = -\frac{1}{12}$$

$$z = \frac{1}{\det(\mathbf{A})}\begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 0 \\ 4 & -3 & 2 \end{vmatrix} = -\frac{1}{12}(-1) = \frac{1}{12}.$$

The solution is therefore $(x, y, z) = \left(\frac{5}{12}, -\frac{1}{12}, \frac{1}{12}\right)$.

4 $\mathbf{A}$ is not invertible if and only if $\det(\mathbf{A}) = 0$; that is, we need

$$\begin{vmatrix} 7-\lambda & -15 \\ 2 & -4-\lambda \end{vmatrix} = (7-\lambda)(-4-\lambda) - (-15)(2) = \lambda^2 - 3\lambda + 2 = 0,$$

which leads us to conclude that $\mathbf{A}$ is not invertible if and only if $\lambda = 1, 2$.

5 The ranks of row-equivalent matrices are equal, and the rank of a matrix is easy to discern if it is in row-echelon form. Thus the way forward is to perform a succession of elementary row operations on $\mathbf{B}$ to obtain a matrix in just such a form:

$$\mathbf{B} \xrightarrow[-r_1+r_3\to r_3]{-2r_1+r_2\to r_2} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 3 & -2 & 2 & 0 \\ 0 & 3 & -2 & 2 & 0 \end{bmatrix} \xrightarrow[-3r_2+r_3\to r_3]{-r_3+r_4\to r_4} \begin{bmatrix} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}'$$

There are three nonzero rows in the row-echelon matrix $\mathbf{B}'$, so $\text{rank}(\mathbf{B}') = 3$, and therefore $\text{rank}(\mathbf{B}) = 3$ also.

Now we determine the null space of $\mathbf{B}$. By definition

$$\text{Nul}(\mathbf{B}) = \{\mathbf{x} : \mathbf{B}\mathbf{x} = \mathbf{0}\}.$$

Since $\mathbf{B}'$ is row-equivalent to $\mathbf{B}$, the system of equations given by the matrix equation $\mathbf{B}'\mathbf{x} = \mathbf{0}$ is equivalent to the system $\mathbf{B}\mathbf{x} = \mathbf{0}$, which is to say they have the same solution set. In fact we may go further:

$$\mathbf{B}' \xrightarrow[-r_3+r_1\to r_1]{r_3+r_2\to r_2} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{B}''.$$

The system $\mathbf{B}''\mathbf{x} = \mathbf{0}$ has the same solution set as $\mathbf{B}\mathbf{x} = \mathbf{0}$, meaning $\text{Nul}(\mathbf{B}) = \text{Nul}(\mathbf{B}'')$. Unpacking $\mathbf{B}''\mathbf{x} = \mathbf{0}$ gives

$$\begin{cases} x_1 + \phantom{x_2 + x_3 -{}} x_4 - x_5 = 0 \\ \phantom{x_1 +{}} x_2 + \phantom{x_3 - x_4 +{}} 2x_5 = 0 \\ \phantom{x_1 + x_2 +{}} x_3 - x_4 + 3x_5 = 0 \end{cases}$$

Letting $x_4 = s$ and $x_5 = t$, we obtain $x_1 = -s + t$, $x_2 = -2t$, and $x_3 = s - 3t$. Hence

$$\text{Nul}(\mathbf{B}) = \left\{ \begin{bmatrix} -s+t \\ -2t \\ s-3t \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ s \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ -3 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

As for the range of $\mathbf{B}$, it is the set of all linear combinations of the column vectors of $\mathbf{B}$, or in other words the span of the column vectors:

$$\text{Ran}(\mathbf{B}) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} \right\}.$$

6 $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution if and only if $\mathbf{A}$ is invertible, and $\mathbf{A}$ is invertible if and only if $\det(\mathbf{A}) \neq 0$. Now,

$$\det(\mathbf{A}) \neq 0 \quad \Leftrightarrow \quad \begin{vmatrix} 1 & 2 & 0 \\ 5 & 1 & \lambda \\ 1 & -1 & 1 \end{vmatrix} \neq 0 \quad \Leftrightarrow \quad \lambda \neq 3.$$

Thus, $\mathbf{A}\mathbf{x} = \mathbf{b}$ has a unique solution for any $\mathbf{b} = [2 \;\; 7 \;\; \mu]^\top$ if and only if $\lambda \neq 3$, no matter what the value of μ is.

In order to consider the possibility that $\mathbf{A}\mathbf{x} = \mathbf{b}$ has no solution, we must first set $\lambda = 3$. Now we perform elementary row operations on the augmented of the resultant system:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 5 & 1 & 3 & 7 \\ 1 & -1 & 1 & \mu \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & -9 & 3 & -3 \\ 0 & -3 & 1 & \mu - 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & 0 & 2 \\ 0 & -9 & 3 & -3 \\ 0 & 0 & 0 & \mu - 1 \end{array}\right].$$

From this we see that the system will have no solution if $\mu \neq 1$; that is, the system is inconsistent if and only if $\lambda = 3$, $\mu \neq 1$.

Finally, we see that the system will have infinitely many solutions if $\mu = 1$; that is, the system is dependent if and only if $\lambda = 3$ and $\mu = 1$.