Math 260 Exam #2 Key (Fall 2018)

1 Let S be the set. Clearly $S \neq \varnothing$ since $0 - 2(0) = 4(0)$ shows that $[0, 0, 0] \in S$.

Suppose $[x_1, y_1, z_1], [x_2, y_2, z_2] \in S$, so $x_1 - 2y_1 = 4z_1$ and $x_2 - 2y_2 = 4z_2$ hold. Adding these equations and rearranging terms then yields

$$(x_1 + x_2) - 2(y_1 + y_2) = 4(z_1 + z_2),$$

which shows that $[x_1 + x_2, y_1 + y_2, z_1 + z_2] \in S$, and hence $[x_1, y_1, z_1] + [x_2, y_2, z_2] \in S$. Therefore S is closed under addition.

Next, suppose that $[x, y, z] \in S$, so that $x - 2y = 4z$. Let $c \in \mathbb{R}$. Then $c(x - 2y) = c(4z)$, or equivalently $cx - 2(cy) = 4(cz)$, which shows that $[cx, cy, cz] \in S$, and hence $c[x, y, z] \in S$. Therefore S is closed under scalar multiplication. We conclude that S is a subspace of $\mathbb{R}^3$.

2 Since $W = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \cdot \mathbf{v} = 0 \text{ for all } \mathbf{v} \in V\}$, it is clear that $\mathbf{0} \in W$ and thus $W \neq \varnothing$. Let $\mathbf{x}, \mathbf{y} \in W$. For any $\mathbf{v} \in V$ we have

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{v} = \mathbf{x} \cdot \mathbf{v} + \mathbf{y} \cdot \mathbf{v} = 0 + 0 = 0,$$

which shows that $\mathbf{x} + \mathbf{y} \in W$. Hence W is closed under addition. Also, for any scalar c we have $(c\mathbf{x}) \cdot \mathbf{v} = c(\mathbf{x} \cdot \mathbf{v}) = (c)(0) = 0$, which shows that $c\mathbf{x}$ and thus W is closed under scalar multiplication.

3 Suppose that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is a system:

$$\begin{cases} c_1 + c_2 - c_3 = 0 \\ 2c_1 + 3c_2 + c_3 = 0 \\ -c_2 + c_3 = 0 \end{cases}$$

The only solution (the derivation is omitted here) is $c_1 = c_2 = c_3 = 0$. Therefore the vectors are linearly independent.

4 Fix $\mathbf{x}, \mathbf{y} \in S$. Let $\mathbf{u} \in [\mathbf{x}, \mathbf{y}]$, so $\mathbf{u} = (1 - s)\mathbf{x} + s\mathbf{y}$ for some $0 \leq s \leq 0$. Now, $\mathbf{x} \in S$ implies $\mathbf{x} = t_1\mathbf{v}_1 + t_2\mathbf{v}_2$ for some $0 \leq t_1, t_2 \leq 1$, and $\mathbf{y} \in S$ implies $\mathbf{y} = \hat{t}_1\mathbf{v}_1 + \hat{t}_2\mathbf{v}_2$ for some $0 \leq \hat{t}_1, \hat{t}_2 \leq 1$. Hence

$$\mathbf{u} = (1 - s)(t_1\mathbf{v}_1 + t_2\mathbf{v}_2) + s(\hat{t}_1\mathbf{v}_1 + \hat{t}_2\mathbf{v}_2) = \big[(1 - s)t_1 + s\hat{t}_1\big]\mathbf{v}_1 + \big[(1 - s)t_2 + s\hat{t}_2\big]\mathbf{v}_2.$$

Now, for $k \in \{1, 2\}$,

$$0 \leq (1 - s)t_k + s\hat{t}_k \leq (1 - s)(1) + (s)(1) = 1,$$

and therefore $\mathbf{u} \in S$. It follows that $[\mathbf{x}, \mathbf{y}] \subseteq S$, and therefore S is convex.

5 Coordinates are $[c_1, c_2, c_3]^\top$, where

$$c_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

A system arises:

$$\begin{cases} \phantom{-c_1 + {}} c_2 + c_3 = 1 \\ c_1 + c_2 \phantom{{}+2c_3} = 1 \\ -c_1 \phantom{{}+c_2} + 2c_3 = 1 \end{cases}$$

The solution to the system is $[c_1, c_2, c_3]^\top = [1, 0, 1]^\top$.

6 Suppose that $\sum_{i=1}^m c_i \mathbf{v}_i = \mathbf{0}$. Then for each $1 \le j \le m$, noting $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for all $i \ne j$, we find that

$$\left(\sum_{i=1}^m c_i \mathbf{v}_i \right) \cdot \mathbf{v}_j = \mathbf{0} \cdot \mathbf{v}_j \quad \Rightarrow \quad \sum_{i=1}^m c_i (\mathbf{v}_i \cdot \mathbf{v}_j) = 0 \quad \Rightarrow \quad c_j (\mathbf{v}_j \cdot \mathbf{v}_j) = 0 \quad \Rightarrow \quad c_j \|\mathbf{v}_j\|^2 = 0.$$

Now, $\|\mathbf{v}_j\| \ne 0$ since $\mathbf{v}_j \ne \mathbf{0}$, and therefore $c_j \|\mathbf{v}_j\|^2 = 0$ implies $c_j = 0$. Therefore the vectors are linearly independent.

7 We have, replacing x and z with s and t along the way,

$$P = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : y = 2x + 3z \right\} = \left\{ \begin{bmatrix} s \\ 2s + 3t \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

Thus a basis for P is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

8 The dimension of this vector space, which we'll denote by S_n, will equal the number of entries in an $n \times n$ matrix that lie on or above the main diagonal. The number of diagonal entries is n, and if the number of entries above the diagonal is k, then the total number of entries is $2k + n$ (the number of entries above, on, and below the diagonal). But we also know the total number of entries is n^2, so $n^2 = 2k + n$, and hence $k = (n^2 - n)/2$. Finally we have

$$\dim(S_n) = n + \frac{n^2 - n}{2} = \frac{n^2 + n}{2}.$$

9 We get a matrix in row-echelon form using elementary row operations on the given matrix:

$$\begin{bmatrix} 1 & 2 & -3 \\ -1 & -2 & 3 \\ 4 & 8 & -12 \\ 1 & -1 & 5 \end{bmatrix} \xrightarrow[-r_1 + r_4 \to r_4]{\substack{-r_1 + r_2 \to r_2 \\ -4r_1 + r_3 \to r_3}} \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -3 & 8 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_4} \begin{bmatrix} 1 & 2 & -3 \\ 0 & -3 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now it is clear that the rank of the original matrix is 2.

10 We have $T(x, y) = (u, v)$ with $u = 3x$ and $v = 7y$. Now,

$$x^2 + y^2 = 1 \quad \Rightarrow \quad \left(\frac{u}{3}\right)^2 + \left(\frac{v}{7}\right)^2 = 1 \quad \Rightarrow \quad \frac{u^2}{9} + \frac{v^2}{49} = 1.$$

The image of the circle $x^2 + y^2 = 1$ under T is therefore the set

$$\left\{(u, v) : \frac{u^2}{9} + \frac{v^2}{49} = 1\right\}$$

in the uv-plane. This is of course an ellipse with center $(0, 0)$.